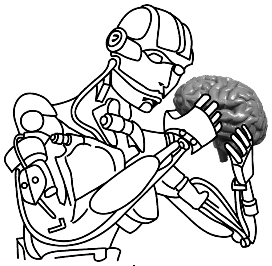
CS545—Contents X

- Lagrange's Method of Deriving Equations of Motion for Rigid Body Systems
    - Lagrange's Equation
    - Generalized Coordinates
    - Potential Energy
    - Kinetic Energy
    - Properties of the Dynamics Equations
- Reading Assignment for Next Class
    - See http://www-clmc.usc.edu/~cs545

Lagrange’s Equations



- The Lagrangian (a potential function) is

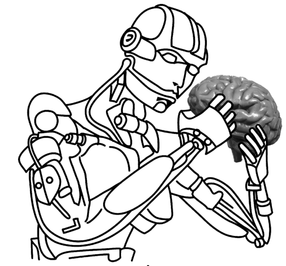
$$L = T - U$$

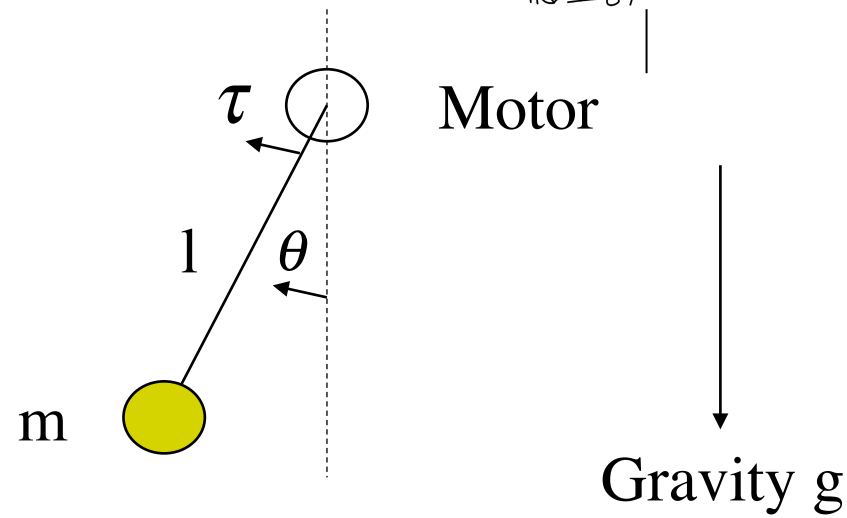
- Lagrange’s Equations are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

- Generalized Coordinates
  - Any set of coordinates that complete describes a dynamical system
  - Lagrangian is coordinate free
- Generalized Forces
  - The corresponding forces or torques applied along the generalized coordinates
  - Includes friction, external, and motor forces (non conservative forces)

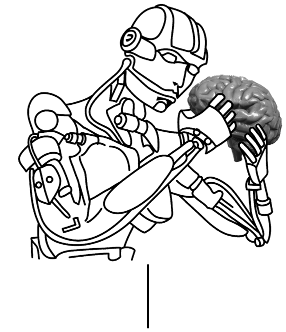
Example: Pendulum





- Kinetic Energy $T = \frac{1}{2} I\dot{\theta}^2$
- Potential Energy $U = mgl(1 - \cos\theta)$
- Lagrangian $L = T - U$
- Equation of Motion $I\ddot{\theta} + mgl\sin(\theta) = \tau$

Kinetic Energy

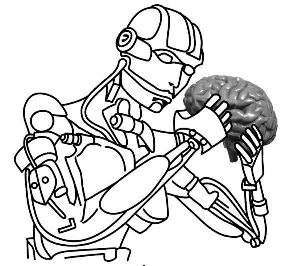


- In General:

$$T = \sum T_i$$

- Problems
  - Velocities of individual links of a robot are not always trivially derived
  - Inertia is configuration dependent
- Note
  - Motors add additional inertial terms, and thus additional term in T and U.

Deriving Kinetic Energies



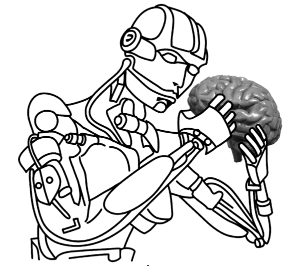
- Velocity of a point can be split into a linear velocity of the center of mass (CM), and an angular velocity relative to CM (S&S, p.134)

$$\dot{p}_i = \dot{p}_{li} + \omega_i \times r_i$$

- Thus, kinetic energy can be split into a linear term of the CM, and an angular term relative to the CM

$$T_i = \frac{1}{2} m_i \dot{p}_{li}^T \dot{p}_{li} + \frac{1}{2} \omega_i^T I_i \omega_i$$

How to calculate the terms in the kinetic Energy?



$$T_i = \frac{1}{2} m_i \dot{p}_{li}^T \dot{p}_{li} + \frac{1}{2} \omega_i^T I_i \omega_i$$

- Note: all these quantities are expressed relative to the world coordinate system
- The mass
  - weigh the link
- The inertia
  - The inertia about the CM in global coordinates can be derived from the inertia of the CM in a local coordinate frame that is fixed to the link as

$$I_i = R_i I_i^i R_i^T$$

    - Where R_i is the rotation matrix from link i-frame to the base.
  - The local inertia can be computed from standard formulae or needs to be found empirically

$\mathbf{p_i}$ is the vector from the origin of the world coordinate system to the origin of the i-th link coordinate system, p is the vector from the origin to the endeffector end, and $\mathbf{z}$ is the i-th joint axis (p.82 S&S)

How to calculate the terms in the kinetic Energy? (cont’d)

- The linear velocity and angular velocity:

$$\begin{bmatrix} \dot{p}_{li} \\ \omega_i \end{bmatrix} = J_p^i \dot{\theta}$$

  - Where J_p^i is the geometric Jacobian for treating the point p as the “endeffector”

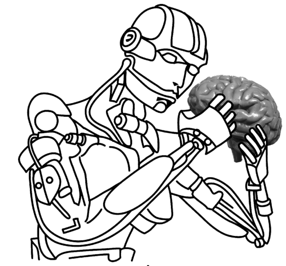
Potential Energy

- In General:

$$U = \sum U_i$$
$$= \sum m_i g p_i(\theta)$$

- Computing the p_i :
  - "truncated direct kinematics"

General Structure of Rigid Body Dynamics Equations



$$\mathbf{B}(\theta)\ddot{\theta} + \mathbf{C}\left(\theta,\dot{\theta}\right) + \mathbf{G}(\theta) = \tau$$

- Note:
  - **B** is the positive definite (generalize) inertia matrix (thus we can compute forward dynamics “easily”)
  - **C** are the centripetal and coriolis forces
  - **G** are the gravitational forces
  - Dynamics parameters are LINEAR in these equations (thus we can do parameter estimation by least squares linear regression)