CS545—Contents III

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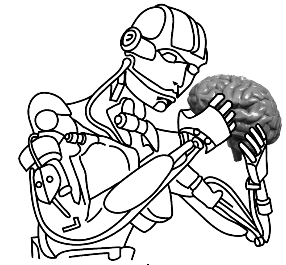
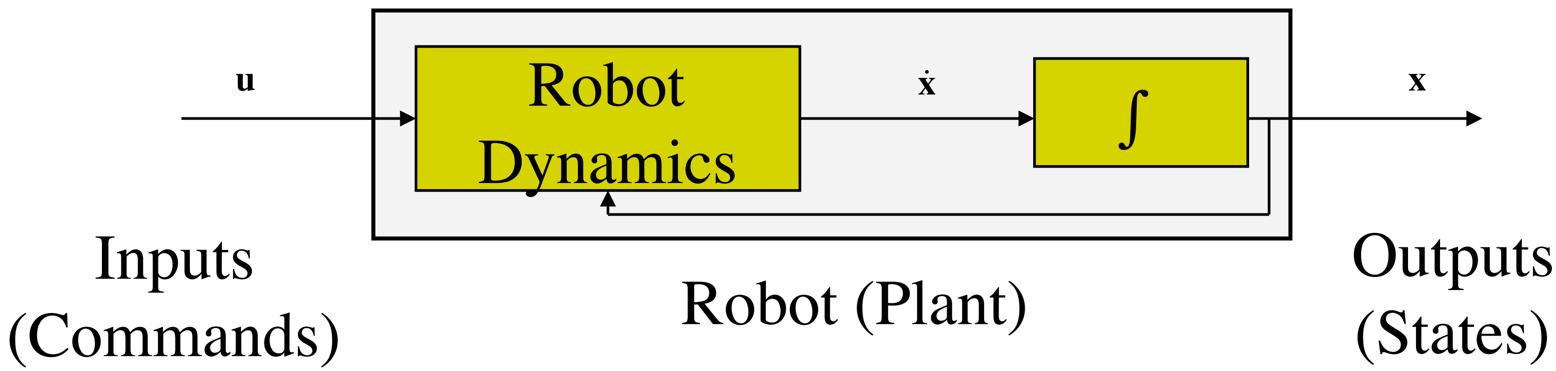
- Reading Assignment for Next Class
  - See http://www-clmc.usc.edu/~cs545

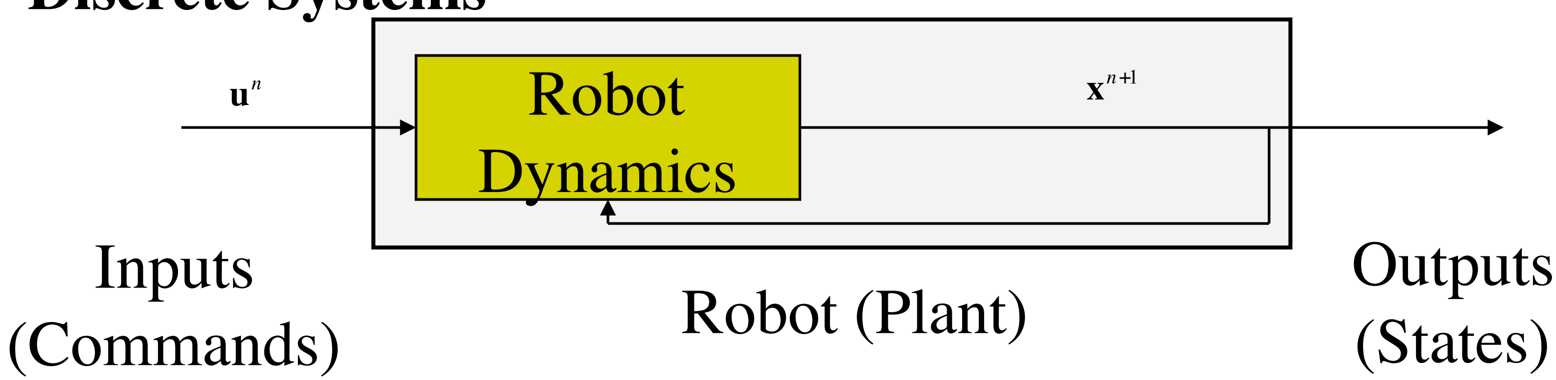
The Plant (Robot)



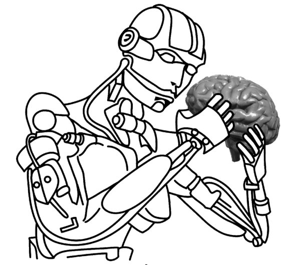
Continuous Systems



Discrete Systems



The Plant (Robot) Model



- Continuous Systems

$$\dot{\mathbf{x}} = f(\mathbf{x},\mathbf{u},t) \leftarrow \text{System dynamics}$$

$$\mathbf{y} = g(\mathbf{x},\mathbf{u},t) \leftarrow \text{Output equations}$$

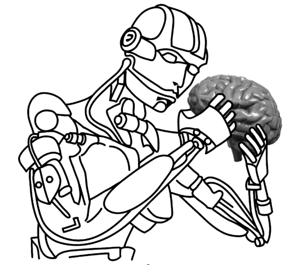
- Discrete Systems

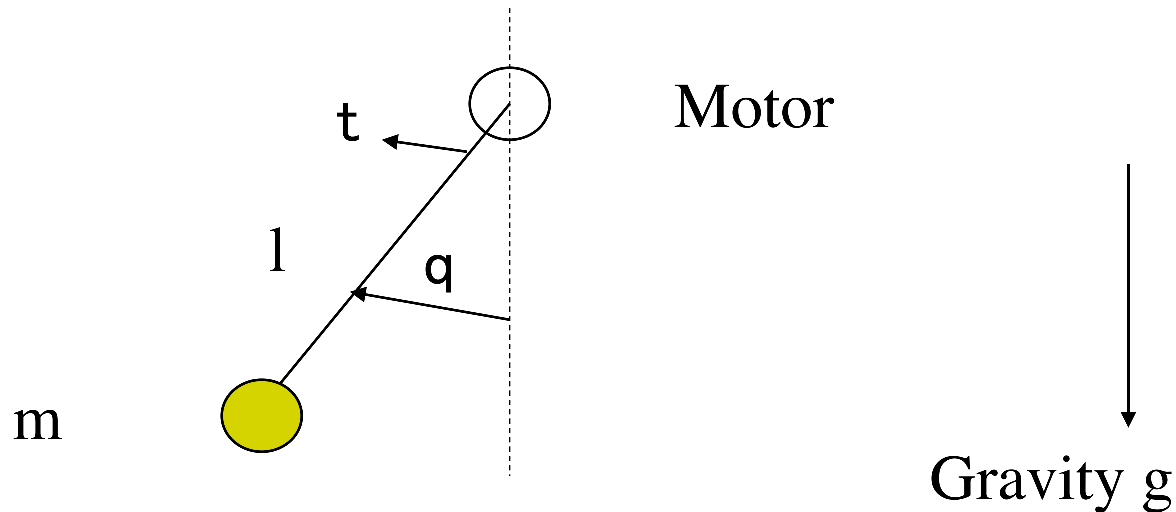
$$\mathbf{x}^{n+1} = f(\mathbf{x}^n,\mathbf{u}^n,n)$$

$$\mathbf{y}^n = g(\mathbf{x}^n,\mathbf{u}^n,n)$$

- Note:
  - If time dependency exists: “time variant” or “nonstationary” or “nonautonomous” system
  - If time dependency does not exist: “time invariant” or “stationary” or “autonomous” system

Example: Pendulum





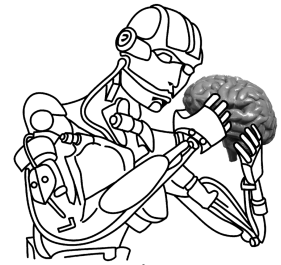
- Assumption:
  - Point mass m
  - No friction
  - External torque motor

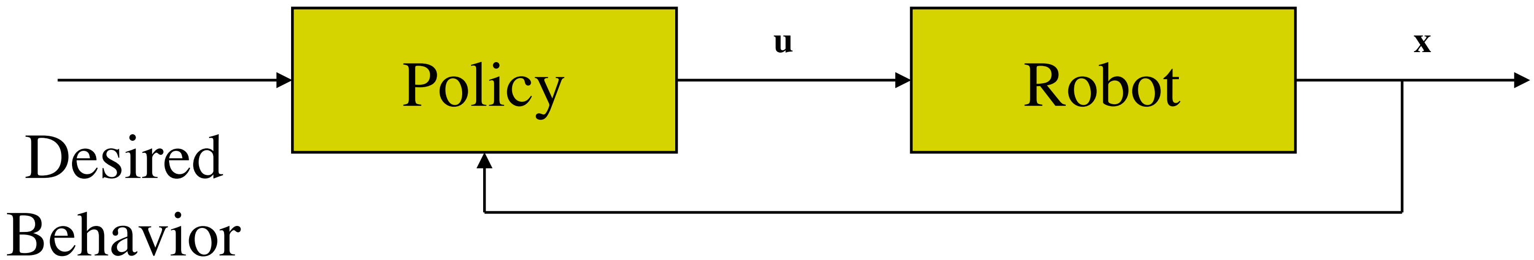
$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

$$ml^2\ddot{\theta} = -mgl\sin(\theta) + \tau$$

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

The Control Policy



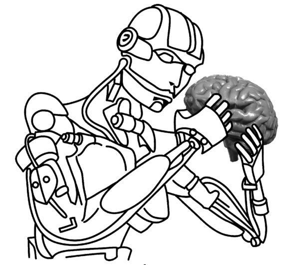


- All what Robotics (Control Theory, AI(?)) is about is to find a “decision making process” that does the right thing at the right time!

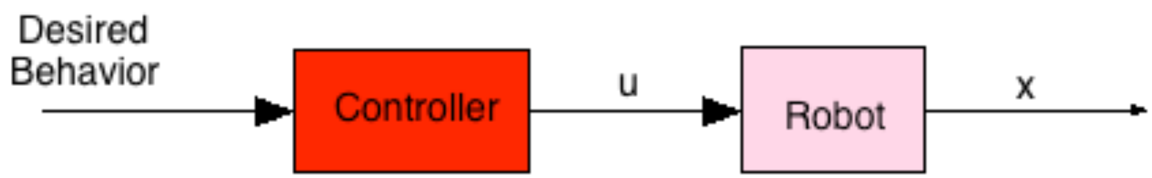
$$\mathbf{u} = \pi(\mathbf{x}, \alpha, t)$$

  - Where a denotes a set of parameters in the policy
- The desired behavior is usually:
  - An external reward
  - An optimization function
  - An explicit desired trajectory

Two Major Control Strategies

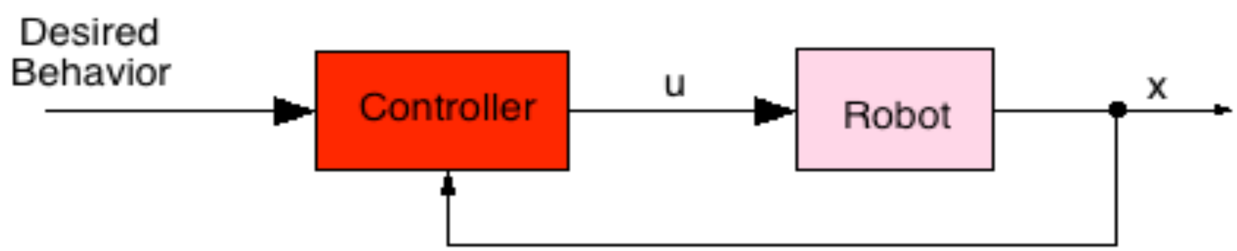


Open Loop Control



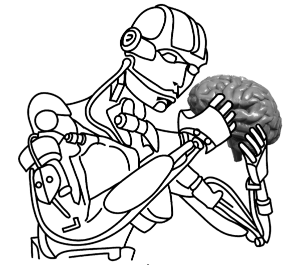
$$\mathbf{u} = \pi(\mathbf{x}, \alpha, t) = \pi(\alpha, t)$$

Closed Loop Control

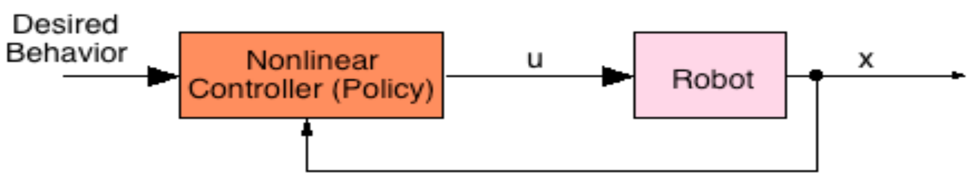


$$\mathbf{u} = \pi(\mathbf{x}, \alpha, t)$$

Types of Feedback Control

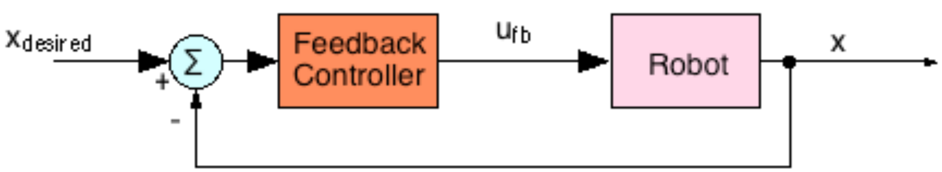


Feedback Control



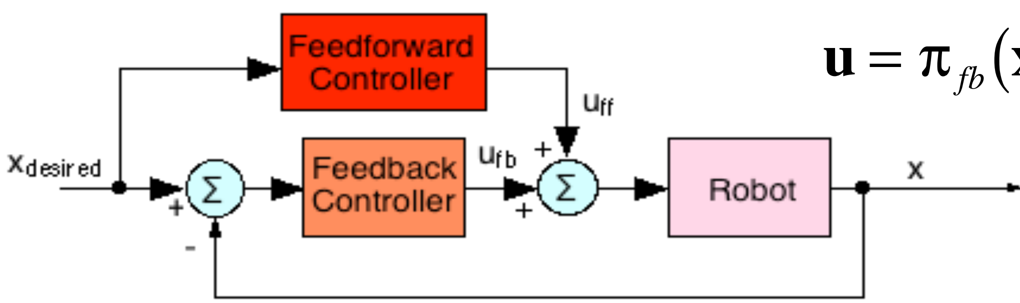
$$\mathbf{u} = \pi(\mathbf{x}, \alpha, t)$$

Negative Feedback Control



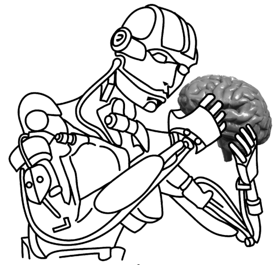
$$\mathbf{u} = \pi(\mathbf{x} - \mathbf{x}_{des}, \alpha, t)$$

Neg. Feedback & Feedforward Control



$$\mathbf{u} = \pi_{fb}(\mathbf{x} - \mathbf{x}_{des}, \alpha, t) + \pi_{ff}(\mathbf{x}_{des}, \alpha, t)$$

Negative Feedback Control



- Mostly based one linear control (i.e.,the control policy is a linear function)
  - Proportional Control ("Position Error")

$$\mathbf{u}_P = \pi\left(\mathbf{x} - \mathbf{x}_{des}, \alpha, t\right) = \mathbf{K}_P\left(\mathbf{x}_{des}(t) - \mathbf{x}(t)\right)$$

  - Derivative Control ("Damping")

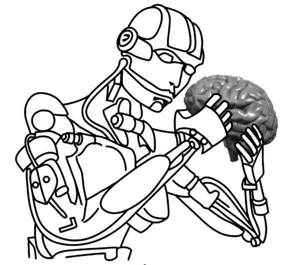
$$\mathbf{u}_D = \pi\left(\mathbf{x} - \mathbf{x}_{des}, \alpha, t\right) = \mathbf{K}_D\left(\dot{\mathbf{x}}_{des}(t) - \dot{\mathbf{x}}(t)\right)$$

  - Integral Control ("Steady State Error")

Note: Usually only based on position errors

$$\mathbf{u}_I(t) = \mathbf{K}_I \int_{\tau=0}^{\tau=t} \left(\mathbf{x}_{des}(t) - \mathbf{x}(t)\right) dt$$

Pendulum with PD Control



$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

variable substitution:

$$x_1 = \dot{\theta}, \quad x_2 = \theta$$

then

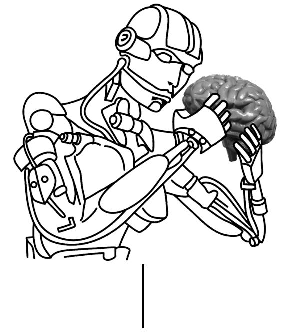
$$\begin{aligned} \dot{x}_1 &= -\frac{g}{l}\sin(x_2) + \frac{\tau}{ml^2} \\ \dot{x}_2 &= x_1 \end{aligned} \quad \text{or} \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix} + \tau \begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix}$$

- Assume the desired position: $x_1 = 0, \quad x_2 = x_d$ and a PD controller output: $u = \tau = k_P(x_d - x_2) + k_D(0 - x_1)$
- At which position does the system come to rest (equilibrium)?

$$0 = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix} + \left(k_P(x_d - x_2) + k_D(0 - x_1)\right)\begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 = 0; \quad \frac{k_P(x_d - x_2)}{ml^2} - \frac{g}{l}\sin(x_2) = 0$$

Pendulum with PD Control (cont'd)



- How to find the equilibrium point?
  - Graphical
  - Approximation by linearization

- The linearized system:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-\frac{g}{l}\sin(x_2)\\ x_1\end{pmatrix}+\tau\begin{pmatrix}1/ml^2\\ 0\end{pmatrix}\xrightarrow[linearization]{}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-\frac{g}{l}x_2\\ x_1\end{pmatrix}+\tau\begin{pmatrix}1/ml^2\\ 0\end{pmatrix} \text{ or}$$

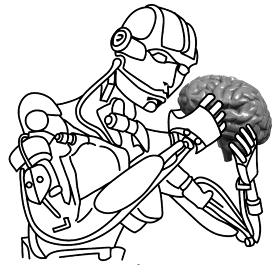
$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-\frac{g}{l}x_2\\ x_1\end{pmatrix}+\tau\begin{pmatrix}1/ml^2\\ 0\end{pmatrix}=\begin{pmatrix}0 & -\frac{g}{l}\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\tau\begin{pmatrix}1/ml^2\\ 0\end{pmatrix} \text{ or}$$

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}$$

- Approximate equilibrium point:

$$x_1=0;\quad \frac{k_P(x_d-x_2)}{ml^2}-\frac{g}{l}x_2=0 \;\Rightarrow\; x_2=\frac{k_P x_d}{k_P+gml}$$

Pendulum with PID Control



- Linearized System

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{g}{l} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \tau \begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix}$$

- The Integral Controller introduces a new state:

$$\dot{x}_3 = k_I \left(x_d - x_2 \right)$$

- The new (linearized) system becomes;

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{g}{l} & 0 \\ 1 & 0 & 0 \\ 0 & -k_p & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \tau/ml^2 \\ 0 \\ k_i x_d \end{pmatrix} \qquad u = \tau = k_P \left(x_d - x_2 \right) + k_D \left(0 - x_1 \right) + k_I x_3$$

This leads to an equilibrium point where all states are 0.