CHAPTER

TWO

STATE-SPACE REPRESENTATION OF DYNAMIC SYSTEMS

Control Systems Design
Friedland, B.
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2.1 MATHEMATICAL MODELS

The most important task confronting the control system analyst is developing a mathematical model of the process of interest. In many situations the essence of the analytical design problem is in the modeling: once that is done the rest of the analysis falls quickly into place.

The control system engineer is often required to deal with a system having a number of subsystems the physical principles of which depend on entirely different types of physical laws. A chemical process, for example, may comprise a chemical reactor, the dynamics of which are the subject of chemical kinetic theory, a heat exchanger which is governed by thermodynamic principles, and various valves and motors the dynamics of which depend on the physics of mechanical and electrical systems. The control of a typical aircraft entails an understanding of the interaction between the airframe governed by principles of aerodynamics and structural dynamics, the actuators which are frequently hydraulic or electrical, and the sensors (gyroscopes and accelerometers) which operate under laws of rigid body dynamics. And, if the human pilot of the aircraft is to be considered, aspects of physiology and psychology enter into the analysis.

One of the attractions of control system engineering is its interdisciplinary content. The control system engineer sees the "big picture" in the challenge to harmonize the operation of a number of interconnected subsystems, each of which operates under a different set of laws. But at the same time the control system engineer is almost totally dependent on the other disciplines. It is simply impossible to gain a sufficient understanding of the details of each of the

subsystems in a typical control process without the assistance of individuals having an intimate understanding of these subsystems. These individuals often have the knowledge that the control system analyst requires, but are not accustomed to expressing it in the form that the analyst would like to have it. The analyst must be able to translate the information he receives from others into the form he needs for his work.

The analyst needs *mathematical models* of the processes in the system under study: equations and formulas that predict how the various devices will behave in response to the inputs to these devices. From the viewpoint of the systems analyst each device is the proverbial "black box," whose operation is governed by appropriate mathematical models. The behavior of the overall process is studied and controlled by studying the interaction of these black boxes.

There are two modeling and analysis approaches in customary use for linear systems: the transfer-function or frequency-domain approach, to be discussed in Chap. 4, and the state-space approach which is the subject of the present chapter.

The feature of the state-space approach that sets it apart from the frequency-domain approach is the representation of the processes under examination by systems of first-order differential equations. This method of representation may appear novel to the engineer who has become accustomed to thinking in terms of transfer functions, but it is not at all a new way of looking at dynamic systems. The state-space is the mode of representation of a dynamic system that would be most natural to the mathematician or the physicist. Were it not that much of classical control theory was developed by electrical engineers, it is arguable that the state-space approach would have been in use much sooner.

State-space methods were introduced to the United States engineering community through the efforts of a small number of mathematically oriented engineers and applied mathematicians during the late 1950s and early 1960s. The spiritual father of much of this activity was Professor Solomon Lefschetz who organized a mathematical systems research group at the Research Institute of Advanced Studies (RIAS) in Baltimore, Md. Lefschetz, already a world-famous mathematician, brought together a number of exceptionally talented engineers and mathematicians committed to the development of mathematical control theory. At Columbia University another group, under the aegis of Professor J. R. Ragazzini, and including R. E. Kalman and J. E. Bertram among others, was also at work developing the foundations of modern control theory.

In the Soviet Union there was less of an emphasis on transfer functions than on differential equations. Accordingly, many of the earliest uses of state-space methods were made by investigators in the Soviet Union. Much of the activity in the United States during the late 1950s entailed translation of the latest Russian papers into English. The Moscow location of the First Congress of the International Federation of Automatic Control (IFAC) in 1960 was entirely appropriate, and provided the first major opportunity for investigators from all over the world to meet and exchange ideas. Although the IFAC congress was concerned with components and applications as well as with control theory, much of the interest of the meeting was on the newest theoretical developments.

2.2 PHYSICAL NOTION OF SYSTEM STATE

The notion of the state of a dynamic system is a fundamental notion in physics. The basic premise of newtonian dynamics is that the future evolution of a dynamic process is entirely determined by its present state. Indeed we might consider this premise as the basis of an abstract definition of the state of a dynamic system:

> The state of a dynamic system is a set of physical quantities, the specification of which (in the absence of external excitation) completely determines the evolution of the system.

The difficulty with this definition, as well as its major advantage, is that the *specific* physical quantities that define the system state are not unique, although their number (called the system *order*) is unique. In many situations there is an obvious choice of the variables (*state variables*) to define the system state, but there are also many cases in which the choice of state variables is by no means obvious.

Newton invented calculus as a means of characterizing the behavior of dynamic systems, and his method continues in use to this very day. In particular, behavior of dynamic systems is represented by systems of ordinary differential equations. The differential equations are said to constitute a *mathematical model* of the physical process. We can predict how the physical process will behave by solving the differential equations that are used to model the process.

In order to obtain a solution to a system of ordinary differential equations, it is necessary to specify a set of *initial conditions.* The number of initial conditions that must be specified defines the *order* of the system. When the differential equations constitute the mathematical model of a physical system, the initial conditions needed to solve the differential equations correspond to physical quantities needed to predict the future behavior of the system. It thus follows that the initial conditions and physical state variables are equal in number.

In analysis of dynamic systems such as mechanical systems, electric networks, etc. the differential equations typically relate the dynamic variables and their time derivatives of various orders. In the state-space approach, all the differential equations in the mathematical model of a system are *first-order* equations: only the dynamic variables and their first derivatives (with respect to time) appear in the differential equations. Since only one initial condition is needed to specify the solution of a first-order equation, it follows that the

number of first-order differential equations in the mathematical model is equal to the order of the corresponding system.

The dynamic variables that appear in the system of first-order equations are called the *state variables.* From the foregoing discussion, it should be clear that the *number* of state variables in the model of a physical process is unique, although the identity of these variables may not be unique. A few familiar examples serve to illustrate these points.

Example 2A Mass acted upon by friction and spring forces The mechanical system consisting of a mass which is acted upon by the forces of friction and a spring is a paradigm of a second-order dynamic process which one encounters time and again in control processes.

Consider an object of mass M moving in a line. In accordance with Newton's law of motion, the acceleration of the object is the total force f acting on the object divided by the mass.

$$\frac{d^2x}{dt^2} = \frac{f}{M} \tag{2A.1}$$

where the direction of f is in the direction of x. We assume that the force f is the sum of two forces, namely a friction force f_1 and a spring force f_2. Both of these forces physically tend to resist the motion of the object. The friction force tends to resist the velocity: there is no friction force unless the velocity is nonzero. The spring force, on the other hand, is proportional to the amount that the spring has been compressed, which is equal to the amount that the object has been displaced. Thus

$$f = f_1 + f_2$$

where

$$f_1 = -\beta\left(\frac{dx}{dt}\right)$$

$$f_2 = -\kappa(x)$$

Thus

$$\frac{d^2x}{dt^2} = -\left[\beta\left(\frac{dx}{dt}\right) + \kappa(x)\right] \Big/ M \tag{2A.2}$$

A more familiar form of (2A.2) is the second-order differential equation

$$M\frac{d^2x}{dt^2} + \beta\left(\frac{dx}{dt}\right) + \kappa(x) = 0 \tag{2A.3}$$

But (2A.2) is a form more appropriate for the state-space representation. Differential equation (2A.2) or its equivalent (2A.3) is a second-order differential equation and its solution requires two initial conditions: x_0, the initial position, and $\dot{x}_0$, the initial velocity.

To obtain a state-space representation, we need two state variables in terms of which the dynamics of (2A.2) can be expressed as two first-order differential equations. The obvious choice of variables in this case are the displacement x and the velocity $v = dx/dt$. The two first-order equations for the process in this case are the equation by which velocity is defined

$$\frac{dx}{dt} = v \tag{2A.4}$$

and (2A.2) expressed in terms of x and v. Since $d^2x/dt^2 = dv/dt$, (2A.2) becomes

$$\frac{dv}{dt} = -[\beta(v) + \kappa(x)]/M \tag{2A.5}$$

Thus (2A.4) and (2A.5) constitute a system of two first-order differential equations in terms of the state variables x and v.

If we wish to control the motion of the object we would include an additional force f_0 external to the system which would be added to the right-hand side of (2A.5)

$$\frac{dv}{dt} = -[\beta(v) + \kappa(x)]/M + f_0/M \tag{2A.6}$$

How such a control force would be produced is a matter of concern to the control system designer. But it is not considered in the present example.

In a practical system both the friction force and the spring force are nonlinear functions of their respective variables and a realistic prediction of the system behavior would entail solution of (2A.4) and (2A.5) in which $\beta(v)$ and $\kappa(x)$ are *nonlinear* functions of their arguments. As an approximation, however, it may be permissible to treat these functions as being linear

$$\beta\left(\frac{dx}{dt}\right) = B\frac{dx}{dt}$$

$$\kappa(x) = Kx$$

where B and K are constants. Often $\beta(\ )$ and $\kappa(\ )$ are treated as linear functions for purposes of control system design, but the accurate nonlinear functions are used in evaluating how the design performs.

A block diagram representation of the differential equations (2A.4) and (2A.6), in accordance with the discussion of Sec. 2.3, is shown in Fig. 2.1.

Example 2B Electric motor with inertia load One of the most common uses of feedback control is to position an inertia load using an electric motor. (See Fig. 2.2.) The inertia load may consist of a very large, massive object such as a radar antenna or a small object such as a precision instrument. An important aspect of the control system design is the selection of a suitable motor, capable of achieving the desired dynamic response and suited to the objective in cost, size, weight, etc. An electric motor is a device that converts electrical energy (input) to mechanical energy (output). The electro-mechanical energy transducer relations are idealizations of Faraday's law of induction and Ampere's law for the force produced on a conductor moving in a magnetic field. In particular, under ideal circumstances the torque developed at the shaft of a motor is proportional to the input current to the motor; the induced emf v ("back emf") is proportional to the speed ω of rotation

$$\tau = K_1 i \tag{2B.1}$$

$$v = K_2\omega \tag{2B.2}$$

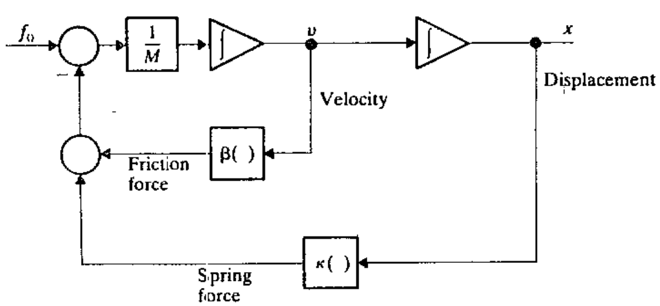

Figure 2.1 Block diagram representing motion of mass with friction and spring reaction forces.

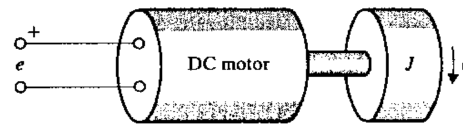

Figure 2.2 DC motor driving inertia load.

The electrical power p_e input to the motor is the product of the current and the induced emf

$$p_e = vi = K_2\omega\tau/K_1 \tag{2B.3}$$

The mechanical output power is the product of the torque and the angular velocity

$$p_m = \omega\tau$$

Thus, from (2B.3)

$$p_e = \frac{K_2}{K_1}p_m$$

If the energy conversion is 100 percent efficient, then

$$K_1 = K_2 = K$$

If the energy-conversion efficiency is less than 100 percent then $K_2/K_1 > 1$.

To completely specify the behavior of the system we need the relationships between the input voltage e and the induced emf, and between the torque and the angular velocity of the motor. These are given by

$$e - v = Ri \qquad \text{(Ohm's law)} \tag{2B.4}$$

where R is the electrical resistance of the motor armature, and

$$\tau = J\frac{d\omega}{dt} \tag{2B.5}$$

where J is the inertia of the load. From (2B.1), (2B.5), and (2B.4)

$$J\frac{d\omega}{dt} = K_1 i = \frac{K_1}{R}(e - v) \tag{2B.6}$$

On using (2B.2) this becomes

$$J\frac{d\omega}{dt} = \frac{K_1}{R}e - \frac{K_1K_2}{R}\omega$$

or

$$\frac{d\omega}{dt} = -\frac{K_1K_2}{JR}\omega + \frac{K_1}{JR}e \tag{2B.7}$$

which is a first-order equation with the angular velocity ω as the state variable and with e serving as the external control input.

The first-order model of (2B.7) is suitable for control of the speed of the shaft rotation. When the position θ of the shaft carrying the inertia J is also of concern, we must add the differential equation

$$\frac{d\theta}{dt} = \omega \tag{2B.8}$$

This and (2B.7) together constitute a second-order system.

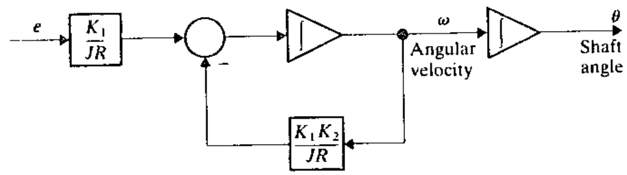

Figure 2.3 Block diagram representing dynamics of dc motor driving inertia load.

Equations (2B.7) and (2B.8) can be arranged in the vector-matrix form

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -K_1K_2/JR\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix} + \begin{bmatrix}0\\ K_1/JR\end{bmatrix}e$$

A block-diagram representation of the differential equations that represent this system is given in Fig. 2.3.

Example 2C Electrical network and its thermal analog It is not generally required to design feedback control systems for electrical networks comprising resistors, capacitors, and inductors. But such networks often are mathematically analogous to mechanical systems which one does desire to control, and an engineer experienced in the analysis of electrical networks might be more comfortable with the latter than with the mechanical systems they represent.

One class of mechanical system which is analogous to an electrical network is a thermal conduction system. Electrical voltages are analogous to temperatures and currents are analogous to heat flow rates. The paths of conduction of heat between various points in the system are represented by resistors; the mass storage of heat in various bodies is represented by capacitances; the input of heat by current sources; and fixed temperatures at the boundaries of the system by voltage sources.

Table 2C.1 summarizes the thermal quantities and their electrical analogs.

As an illustration of the use of electrical analogs of thermal systems, consider the system shown in Fig. 2.4 consisting of two masses of temperatures T_1 and T_2 embedded in a thermally insulating medium contained in a metal container which, because of its high thermal conductivity, may be assumed to have a constant temperature T_0. The temperatures T_1 and T_2 of the masses are to be controlled by controlling the temperature T_0 of the container.

Table 2C.1 Electrical analogs of thermal systems

Thermal system			Electrical system		
Quantity	Symbol	Unit	Quantity	Symbol	Unit
Temperature	T	deg	Voltage	v	volt
Heat flux	q	cal/s	Current	i	ampere
Thermal resistivity	R	deg · s/cal	Resistance	R	ohm
Thermal capacity	C	cal/deg	Capacitance	C	farad
Conduction equation	$q = \frac{1}{R}(T_2 - T_1)$			$i = \frac{1}{R}(v_2 - v_1)$	
Storage equation	$\frac{dT}{dt} = \frac{q}{C}$			$\frac{dv}{dt} = \frac{1}{C}i$	

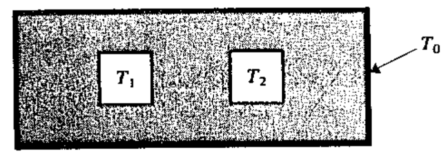

Figure 2.4 Thermal system with two capacitances.

An electrical analog of the system is shown in Fig. 2.5. The capacitors C_1 and C_2 represent the heat capacities of the masses; the resistor R_3 represents the path of heat flow from mass 1 to mass 2; R_1 and R_2 represent the heat flow path from these masses to the metal container.

The differential equations governing the thermal dynamics of the mechanical system are the same as the differential equations of the electrical system, which can be obtained by various standard methods. By use of nodal analysis, for example, it is determined that

$$C_1\frac{dv_1}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_3}\right)v_1 - \frac{1}{R_3}v_2 - \frac{1}{R_1}e_0 = 0$$
$$C_2\frac{dv_2}{dt} + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)v_2 - \frac{1}{R_3}v_1 - \frac{1}{R_2}e_0 = 0 \qquad (2C.1)$$

The appropriate state variables for the process are the capacitor voltages v_1 and v_2. The temperature of the case is represented by a voltage source e_0 which is the input variable to the process. Thus the differential equations of the process are

$$\frac{dv_1}{dt} = -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_3}\right)v_1 + \frac{1}{C_1R_3}v_2 + \frac{1}{R_1C_1}e_0$$
$$\frac{dv_2}{dt} = \frac{1}{C_2R_3}v_1 - \frac{1}{C_2}\left(\frac{1}{R_2} + \frac{1}{R_3}\right)v_2 + \frac{1}{R_2C_2}e_0 \qquad (2C.2)$$

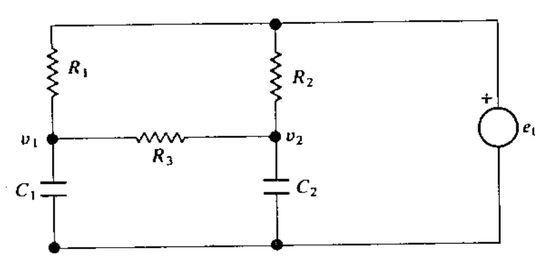

Figure 2.5 Electrical analog of thermal system of Fig. 2.4.

The foregoing examples are typical of the general form of the dynamic equations of a dynamic process. The state variables of a process of order k are designated by $x_1, x_2, \ldots, x_k$ and the external inputs by $u_1, u_2, \ldots, u_l$

$$\begin{aligned}
\dot{x}_1 &= \frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_k, u_1, u_2, \ldots, u_l, t) \\
\dot{x}_2 &= \frac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_k, u_1, u_2, \ldots, u_l, t) \\
&\cdots\cdots\cdots\cdots\cdots \\
\dot{x}_k &= \frac{dx_k}{dt} = f_k(x_1, x_2, \ldots, x_k, u_1, u_2, \ldots, u_l, t)
\end{aligned} \tag{2.1}$$

These equations express the time-derivatives of each of the state variables as general functions of all the state variables, inputs, and (possibly) time. The dot over a variable is Newton's notation for the derivative with respect to time.

To simplify the notation the state variables $x_1, x_2, \ldots, x_k$ and control variables $u_1, u_2, \ldots, u_l$ are collected in vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_l \end{bmatrix} \tag{2.2}$$

called the *state vector* and the *input vector*, respectively. These are vectors in the mathematical sense and not necessarily in the physical sense. The components of a *physical* vector are usually projections of a physical quantity (e.g., force, velocity) along a set of reference axes. But the components of the state vector of a dynamic system generally do not have this interpretation and need not even represent the same kind of physical quantities: As our examples show, position and velocity are typical components of a mathematical state vector.

In some books the state vector is printed in a special typeface such as boldface **x**, to distinguish it from a scalar variable x. We have chosen not to use any special typeface for the state vector since there is rarely any possibility of confusing the entire state vector x with one of its components x_i (always written with a subscript). In subsequent chapters we will make use of a boldface symbol **x** to denote the *metastate* of a system, which is the vector comprising the state (or error) vector, concatenated with the exogenous state vector x_0 as explained in Chap. 5 and later.

Using vector notation, the set of differential equations (2.1) that defines a general process can be written compactly as the single vector differential equation

$$\dot{x} = \frac{dx}{dt} = f(x, u, t) \tag{2.3}$$

where $f(x, u, t)$ is understood to be a k-dimensional vector-valued function of $k + l + 1$ arguments. When time t does not appear explicitly in any of the functions f_i in (2.1), i.e., in the vector f of (2.3), the system is said to be *time-invariant*. If (2.3) is an accurate model of a physical process, we would expect it to be time-invariant, since we do not have physical laws that change with time. In many situations, however, the differential equations represented by (2.3) are only an approximate model of the physical world, either because a more accurate model is not known, or because it is too complicated to be useful in the intended application. Very often such approximate models are time-varying.

An exact model of a physical process is usually nonlinear. But fortunately many processes can be adequately approximated by linear models over a significant range of operation. In the state-space model of a linear process, the general differential equations of (2.1) take the special form:

$$\begin{aligned}
\dot{x}_1 &= \frac{dx_1}{dt} = a_{11}(t)x_1 + \cdots + a_{1k}(t)x_k + b_{11}(t)u_1 + \cdots + b_{1l}(t)u_l \\
\dot{x}_2 &= \frac{dx_2}{dt} = a_{21}(t)x_1 + \cdots + a_{2k}(t)x_k + b_{21}(t)u_1 + \cdots + b_{2l}(t)u_l \\
&\cdots\cdots\cdots\cdots\cdots \\
\dot{x}_k &= \frac{dx_k}{dt} = a_{k1}(t)x_1 + \cdots + a_{kk}(t)x_k + b_{k1}(t)u_1 + \cdots + b_{kl}(t)u_l
\end{aligned} \tag{2.4}$$

In vector notation, using the definitions of the state and control vectors as defined in (2.2), the linear dynamic model of (2.4) is written

$$\dot{x} = \frac{dx}{dt} = A(t)x + B(t)u \tag{2.5}$$

where $A(t)$ and $B(t)$ are *matrices* given by

$$A(t) = \begin{bmatrix} a_{11}(t) \cdots a_{1k}(t) \\ a_{21}(t) \cdots a_{2k}(t) \\ \cdots\cdots\cdots \\ a_{k1}(t) \cdots a_{kk}(t) \end{bmatrix} \qquad B(t) = \begin{bmatrix} b_{11}(t) \cdots b_{1l}(t) \\ b_{21}(t) \cdots b_{2l}(t) \\ \cdots\cdots\cdots \\ b_{k1}(t) \cdots b_{kl}(t) \end{bmatrix} \tag{2.6}$$

It is noted that the matrix $A(t)$ is always a square (k by k) matrix, but that the matrix $B(t)$ need not be square. In most processes of interest the number l of inputs is smaller than the number of state variables: $B(t)$ is a tall, thin matrix. Often there is only one input and the matrix $B(t)$ is only one column wide.

When the system is time-invariant, none of the elements in the matrices A and B depend upon time. Most of this book is concerned with linear, time-invariant processes, having the dynamic equations

$$\dot{x} = Ax + Bu \tag{2.7}$$

where A and B are constant matrices.

Although the concept of the state of a system is fundamental, there are many situations in which one is not interested in the state directly, but only in its effect on the system *output* vector $y(t)$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} \tag{2.8}$$

for a system having m outputs. In a linear system the output vector is assumed to be a linear combination of the state and the input

$$y(t) = C(t)x(t) + D(t)u(t) \tag{2.9}$$

where $C(t)$ is an $m \times k$ matrix and $D(t)$ is an $m \times 1$ matrix. If the system is time-invariant, $C(t)$ and $D(t)$ are constant matrices.

The outputs of a system are generally those quantities which can be *observed*, i.e., measured by means of suitable sensors. Accordingly, the output vector is called the *observation vector* and (2.9) is called the observation equation.

The presence of the matrix D in (2.9) means that there is a direct connection between the input $u(t)$ and the output $y(t)$, without the intervention of the state $x(t)$. Although there is no general reason for the matrix D to be absent in a practical application, it turns out that it *is* absent in the overwhelming majority of applications. This is fortunate, because the presence of D increases the complexity of much of the theory. Thus most of our development will rest on the assumption that $D = 0$.

The input vector u in (2.7) represents the assemblage of all physical quantities that affect the behavior of the state. From the control system design standpoint, however, the inputs are of two types:

Control inputs, produced intentionally by the operation of the control system, and
"Exogenous" inputs, present in the environment and not subject to control within the system.

It is customary to reserve the symbol u for the control inputs and to use another symbol for the exogenous inputs. (The word "exogenous," widely used in the field of economics and other social sciences, is gaining currency in the field of control theory.) In this book we shall find it convenient to represent the exogenous inputs by the vector x_0. The use of the letter "x" suggests that the exogenous inputs are state variables and so they may be regarded: x_0 may be

regarded as the *state of the environment.* (Later in the book we shall concatenate the state x of the system to be controlled with the state x_0 of the environment into a metastate of the overall process.)

Thus, separating the input u of (2.7) into a control input and an exogenous input, (2.7) becomes

$$\dot{x} = Ax + Bu + Ex_0 \tag{2.10}$$

which, together with (2.9) will serve as the general representation of a linear system.

2.3 BLOCK-DIAGRAM REPRESENTATIONS

System engineers often find it helpful to visualize the relationships between dynamic variables and subsystems of a system by means of block diagrams. Each subsystem is represented by a geometric figure (such as a rectangle, a circle, a triangle, etc.) and lines with arrows on them show the inputs and the outputs. For many systems, these block diagrams are more expressive than the mathematical equations to which they correspond.

The relationships between the variables in a linear system (2.4) can be expressed using only three kinds of elementary subsystems:

Integrators, represented by triangles
Summers, represented by circles, and
Gain elements, represented by rectangular or square boxes as shown in Fig. 2.6.

An integrator is a block-diagram element whose output is the integral of the input; put in other words, it is the element whose input is the derivative of the output.

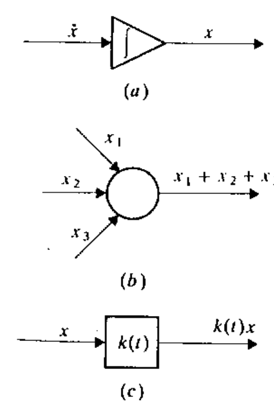

Figure 2.6 Elements used in block-diagram representation of linear systems. (*a*) Integrator; (*b*) Summer; (*c*) Gain element.

A summer is a block-diagram element whose output is the sum of all its inputs.

A gain element is a block-diagram element whose output is proportional to its input. The constant of proportionality, which may be time-varying, is placed inside the box (when space permits) or adjacent to it.

Note that the integrator and the gain element are single-input elements; the summer, on the other hand, always has at least two inputs.

A general block diagram for a second-order system ($k = 2$) with two external inputs u_1 and u_2 is shown in Fig. 2.7. Two integrators are needed, the outputs of which are x_1 and x_2, and the inputs to which are $\dot{x}_1$ and $\dot{x}_2$, respectively. From the general form of the differential equations (2.4) these are given by

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_{11}x_1 + b_{12}x_2$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_{21}x_1 + b_{22}x_2$$

which are the relationships expressed by the outputs of the two summers shown in Fig. 2.7.

The same technique applies in higher-order systems. If the A matrix has many nonzero terms, the diagram can look like a plate of spaghetti and meatballs. In most practical cases, however, the A matrix is fairly sparse, and

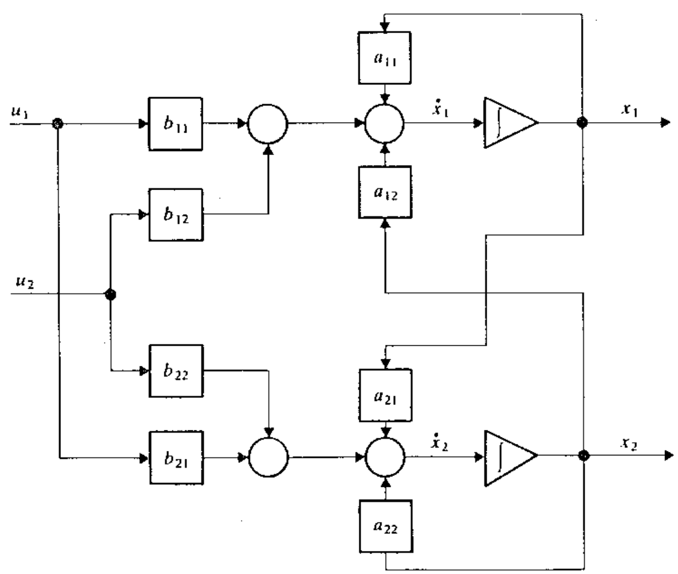

Figure 2.7 Block diagram of general second-order linear system.

with some attention to layout it is possible to draw a block diagram with a minimum of crossed lines.

To simplify the appearance of the block-diagram it is sometimes convenient to use redundant summers. This is shown in Fig. 2.7. Instead of using two summers, one feeding another, in front of each integrator we could have drawn the diagram with only one summer with four inputs in front of each integrator. But the diagram as shown has a neater appearance. Another technique to simplify the appearance of a block diagram is to show a sign reversal by means of a minus sign adjacent to the arrow leading into a summer instead of a gain element with a gain of -1. This usage is illustrated in Figs. 2.1 and 2.3 of the foregoing examples.

Although there are several international standards for block-diagram symbols, these standards are rarely adhered to in technical papers and books. The differences between the symbols used by various authors, however, are not large and are not likely to cause the reader any confusion.

The following examples illustrate the use of matrices and block diagrams to represent the dynamics of various processes.

Often it is convenient to express relationships between vector quantities by means of block diagrams. The block-diagram symbols of Fig. 2.6 can also serve to designate operations on vectors. In particular, when the input to an integrator of Fig. 2.6(a) is a vector quantity, the output is a vector each component of which is the integral of the corresponding input. The summer of Fig. 2.6(b) represents a vector summer, and the gain element box of Fig. 2.6(c) represents a matrix. In the last case, the matrix need not be square and the dimension of the vector of outputs from the box need not equal the dimension of the vector

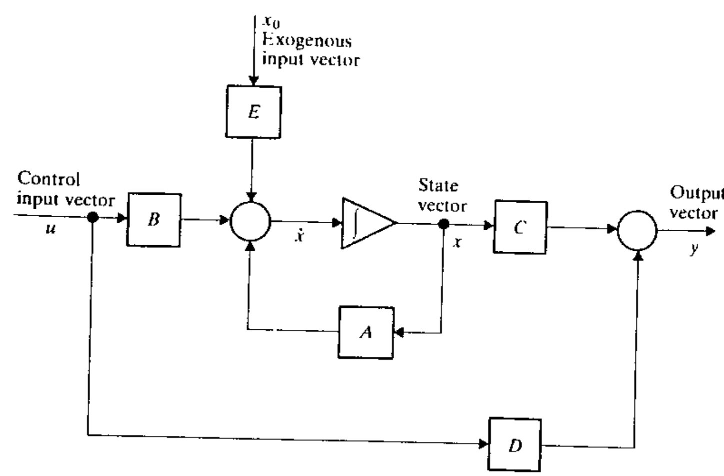

Figure 2.8 Block-diagram representation of general linear system.

of inputs. Using this mode of representation, the block diagram of Fig. 2.8 represents the general system given by (2.9) and (2.10).

Example 2D Hydraulically actuated tank gun turret The control of a hydraulically actuated gun turret in an experimental tank has been studied by Loh, Cheok, and Beck.[1] The linearized dynamic model they used for each axis (elevation, azimuth) is given by

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= p + d_r \\ \dot{p} &= -\Omega_m p + \frac{K_m}{J} q - \frac{K_m}{J} \omega + d_p \\ \dot{q} &= -K_v L_v q - K_v K_{\Delta p} J p + K_v u + d_q\end{aligned} \tag{2D.1}$$

where $x_1 = \theta$ = turret angle
$x_2 = \omega$ = turret angular rate
$x_3 = p$ = angular acceleration produced by hydraulic drive
$x_4 = q$ = hydraulic servo valve displacement
u = control input to servo valve
K_m = servo motor gain
J = turret inertia
Ω_m = motor natural frequency
K_v = servo valve gain
$K_{\Delta p}$ = differential pressure feedback coefficient

The quantities d_r, d_p, and d_q represent disturbances, including effects of nonlinearities not accounted for by the linearized model (2D.1).

With the state variable definitions given above, the matrices of this process are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & K_m/J & -\Omega_m & -K_m/J \\ 0 & 0 & -K_v K_{\Delta p} J & -K_v L_v \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_v \end{bmatrix}$$

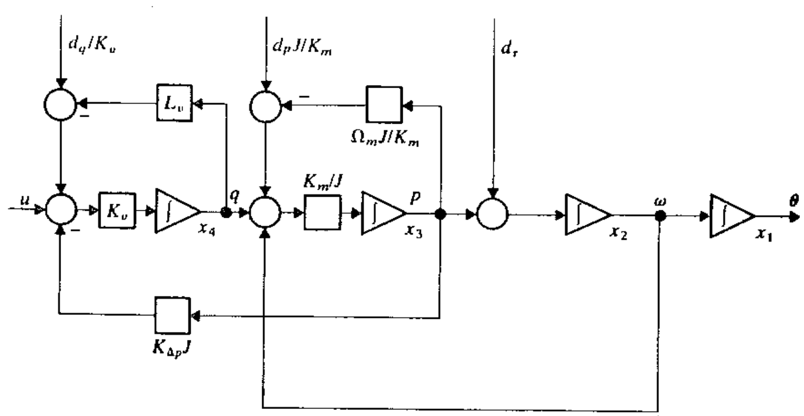

Figure 2.9 Dynamic model of hydraulically actuated tank gun turret.

Table 2D.1 Numerical values of parameters in tank turret control

Parameter	Numerical value: Azimuth	Elevation
K_v	94.3	94.3
L_v	1.00	1.07
J (ft-lb · s²)	7900.	2070.
K_m	8.46×10^6	1.96×10^6
ω_m (rad/s)	45.9	17.3
$K_{\Delta p}$	6.33×10^{-6}	3.86×10^{-5}

Numerical data for a specific tank were found by Loh, Cheok, and Beck to be as given in Table 2D.1

A block-diagram representation of the dynamics represented by (2D.1) is shown in Fig. 2.9.

2.4 LAGRANGE'S EQUATIONS

The equations governing the motion of a complicated mechanical system, such as a robot manipulator, can be expressed very efficiently through the use of a method developed by the eighteenth-century French mathematician Lagrange. The differential equations that result from use of this method are known as *Lagrange's equations* and are derived from Newton's laws of motion in most textbooks on advanced dynamics.[2, 3]

Lagrange's equations are particularly advantageous in that they automatically incorporate the constraints that exist by virtue of the different parts of a system being connected to each other, and thereby eliminate the need for substituting one set of equations into another to eliminate forces and torques of constraint. Since they deal with scalar quantities (potential and kinetic energy) rather than with vectors (forces and torques) they also minimize the need for complicated vector diagrams that are usually required to define and resolve the vector quantities in the proper coordinate system. The advantages of Lagrange's equations may also turn out to be disadvantages, because it is necessary to identify the generalized coordinates correctly at the very beginning of the analysis of a specific system. An error made at this point may result in a set of differential equations that look correct but do not constitute the correct model of the physical system under investigation.

The fundamental principle of Lagrange's equations is the representation of the system by a set of generalized coordinates q_i $(i = 1, 2, \ldots, r)$, one for each independent degree of freedom of the system, which completely incorporate the constraints unique to that system, i.e., the interconnections between the parts of the system. After having defined the generalized coordinates, the kinetic energy T is expressed in terms of these coordinates and their derivatives, and the

CHAPTER

THREE

DYNAMICS OF LINEAR SYSTEMS

3.1 DIFFERENTIAL EQUATIONS REVISITED

In the last chapter we saw that the dynamic behavior of many dynamic systems is quite naturally characterized by systems of first-order differential equations. For a general system these equations in state space notation take the form

$$\dot{x} = f(x, u, t)$$

and in a linear system they take the special form

$$\dot{x} = A(t)x + B(t)u \tag{3.1}$$

where $x = [x_1, x_2, \ldots, x_k]'$ is the system state vector and $u = [u_1, u_2, \ldots, u_m]'$ is the input vector.

If the matrices A and B in (3.1) are constant matrices, i.e., not functions of time, the system is said to be "time-invariant." Time-varying systems are conceptually and computationally more difficult to handle than time-invariant systems. For this reason our attention will be devoted primarily to time-invariant systems. Fortunately many processes of interest can be approximated by linear, time-invariant models.

In using the conventional, frequency-domain approach the differential equations are converted to transfer functions as soon as possible, and the dynamics of a system comprising several subsystems is obtained by combining the transfer functions of the subsystems using well-known techniques (reviewed in Chap. 4). With the state-space methods, on the other hand, the description of the system dynamics in the form of differential equations is retained throughout the analysis and design. In fact, if a subsystem is characterized by a transfer

function it is often necessary to convert the transfer function to differential equations in order to proceed by state-space methods.

In this chapter we shall develop the general formula for the solution of a vector-matrix differential equation in the form of (3.1) in terms of a very important matrix known as the *state-transition matrix* which describes how the state $x(t)$ of the system at some time t evolves into (or from) the state $x(\tau)$ at some other time τ. For time-invariant systems, the state-transition matrix is the matrix exponential function, which is easily calculated. For most time-varying systems, however, the state-transition matrix, although known to exist, cannot be expressed in terms of simple functions (such as real or complex exponentials) or even not-so-simple functions (such as Bessel functions, hypergeometric functions). Thus, while many of the results developed for time-invariant systems apply to time-varying systems, it is very difficult as a practical matter to carry out the required calculations. This is one reason why our attention is confined mainly (but not exclusively) to time-invariant systems. The world of real applications contains enough of the latter to keep a design engineer occupied.

3.2 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS IN STATE-SPACE FORM

Time-invariant dynamics The simplest form of the general differential equation of the form (3.1) is the "homogeneous," i.e., unforced equation

$$\dot{x} = Ax \tag{3.2}$$

where A is a constant k by k matrix. The solution to (3.2) can be expressed as

$$x(t) = e^{At}c \tag{3.3}$$

where e^{At} is the matrix exponential function

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \cdots \tag{3.4}$$

and c is a suitably chosen constant vector. To verify (3.3) calculate the derivative of $x(t)$

$$\frac{dx(t)}{dt} = \frac{d}{dt}(e^{At})c \tag{3.5}$$

and, from the defining series (3.4),

$$\frac{d}{dt}(e^{At}) = A + A^2t + A^3\frac{t^2}{2!} + \cdots = A\left(I + At + A^2\frac{t^2}{2!} + \cdots\right) = A\,e^{At}$$

Thus (3.5) becomes

$$\frac{dx(t)}{dt} = Ae^{At}c = Ax(t)$$

which was to be shown. To evaluate the constant c suppose that at some time τ the state $x(\tau)$ is given. Then, from (3.3),

$$x(\tau) = e^{A\tau}c \tag{3.6}$$

Multiplying both sides of (3.6) by the inverse of $e^{A\tau}$ we find that

$$c = (e^{A\tau})^{-1}x(\tau)$$

Thus the general solution to (3.2) for the state $x(t)$ at time t, given the state $x(\tau)$ at time τ, is

$$x(t) = e^{At}(e^{A\tau})^{-1}x(\tau) \tag{3.7}$$

The following property of the matrix exponential can readily be established by a variety of methods—the easiest perhaps being the use of the series definition (3.4)—

$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2} \tag{3.8}$$

for any t_1 and t_2. From this property it follows that

$$(e^{A\tau})^{-1} = e^{-A\tau} \tag{3.9}$$

and hence that (3.7) can be written

$$x(t) = e^{A(t-\tau)}x(\tau) \tag{3.10}$$

The matrix $e^{A(t-\tau)}$ is a special form of the *state-transition matrix* to be discussed subsequently.

We now turn to the problem of finding a "particular" solution to the nonhomogeneous, or "forced," differential equation (3.1) with A and B being constant matrices. Using the "method of the variation of the constant,"[1] we seek a solution to (3.1) of the form

$$x(t) = e^{At}c(t) \tag{3.11}$$

where $c(t)$ is a function of time to be determined. Take the time derivative of $x(t)$ given by (3.11) and substitute it into (3.1) to obtain:

$$Ae^{At}c(t) + e^{At}\dot{c}(t) = Ae^{At}c(t) + Bu(t)$$

or, upon cancelling the terms $Ae^{At}c(t)$ and premultiplying the remainder by e^{-At},

$$\dot{c}(t) = e^{-At}Bu(t) \tag{3.12}$$

Thus the desired function $c(t)$ can be obtained by simple integration (the mathematician would say "by a quadrature")

$$c(t) = \int_T^t e^{-A\lambda}Bu(\lambda)\,d\lambda$$

The lower limit T on this integral cannot as yet be specified, because we will need to put the particular solution together with the solution to the homogeneous equation to obtain the complete (general) solution. For the present, let T be undefined. Then the particular solution, by (3.11), is

$$x(t) = e^{At}\int_T^t e^{-A\lambda}Bu(\lambda)\,d\lambda = \int_T^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.13}$$

In obtaining the second integral in (3.13), the exponential e^{At}, which does not depend on the variable of integration λ, was moved under the integral, and property (3.8) was invoked to write $e^{At}e^{-A\lambda} = e^{A(t-\lambda)}$.

The complete solution to (3.1) is obtained by adding the "complementary solution" (3.10) to the particular solution (3.13). The result is

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_T^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.14}$$

We can now determine the proper value for lower limit T on the integral. At $t = \tau$ (3.14) becomes

$$x(\tau) = x(\tau) + \int_T^\tau e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.15}$$

Thus, the integral in (3.15) must be zero for any $u(t)$, and this is possible only if $T = \tau$. Thus, finally we have the complete solution to (3.1) when A and B are constant matrices

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.16}$$

This important relation will be used many times in the remainder of the book. It is worthwhile dwelling upon it. We note, first of all, that the solution is the sum of two terms: the first is due to the "initial" state $x(\tau)$ and the second—the integral—is due to the input $u(\tau)$ in the time interval $\tau \leqq \lambda \leqq t$ between the "initial" time τ and the "present" time t. The terms initial and present are enclosed in quotes to denote the fact that these are simply convenient definitions. There is no requirement that $t \geqq \tau$. The relationship is perfectly valid even when $t \leqq \tau$.

Another fact worth noting is that the integral term, due to the input, is a "convolution integral": the contribution to the state $x(t)$ due to the input u is the convolution of u with $e^{At}B$. Thus the function $e^{At}B$ has the role of the impulse response[1] of the system whose output is $x(t)$ and whose input is $u(t)$.

If the output y of the system is not the state x itself but is defined by the observation equation

$$y = Cx$$

then this output is expressed by

$$y(t) = Ce^{A(t-\tau)}x(t) + \int_\tau^t Ce^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.17}$$

and the impulse response of the system with y regarded as the output is $C e^{A(t-\lambda)} B$.

The development leading to (3.16) and (3.17) did not really require that B and C be constant matrices. By retracing the steps in the development it is readily seen that when B and C are time-varying, (3.16) and (3.17) generalize to

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\lambda)}B(\lambda)u(\lambda)\,d\lambda \tag{3.18}$$

and

$$y(t) = C(t)\,e^{A(t-\tau)}x(\tau) + \int_\tau^t C(t)\,e^{A(t-\lambda)}B(\lambda)u(\lambda)\,d\lambda \tag{3.19}$$

Time-varying dynamics Unfortunately, however, the results expressed by (3.18) and (3.19) do *not* hold when A is time-varying.

In any unforced (homogeneous) system the state at time t depends only on the state at time τ. In a linear system, this dependence is linear; thus we can always write the solution to $\dot{x} = A(t)x$ as

$$x(t) = \Phi(t, \tau)x(\tau) \tag{3.20}$$

The matrix $\Phi(t, \tau)$ that relates the state at time t to the state at time τ is generally known as the ***state-transition matrix*** because it defines how the state $x(\tau)$ evolves (or "transitions") into (or from) the state $x(t)$. In a time-invariant system $\Phi(t, \tau) = e^{A(t-\tau)}$, but there is no simple expression for the state-transition matrix in a time-varying system. The absence of such an expression is rarely a serious problem, however. It is usually possible to obtain a control system design from only a knowledge of the dynamics matrix $A(t)$, without having an expression for the transition matrix.

The complete solution to (3.1) can be expressed in the form of (3.18), with the general transition matrix $\Phi(t, \tau)$ replacing the matrix exponential of a time-invariant system. The general solution is thus given by

$$x(t) = \Phi(t, \tau)x(\tau) + \int_\tau^t \Phi(t, \lambda)B(\lambda)u(\lambda)\,d\lambda \tag{3.21}$$

$$y(t) = C(t)\Phi(t, \tau)x(\tau) + \int_\tau^t C(t)\Phi(t, \lambda)B(\lambda)u(\lambda)\,d\lambda \tag{3.22}$$

The derivation of (3.21) follows the same pattern as was used to obtain (3.18). The reader might wish to check his comprehension of the development by deriving (3.21). The development can also be found in a number of textbooks on linear systems, [1] for example.

The state-transition matrix The state-transition matrix for a time-invariant system can be calculated by various methods. One of these is to use the series definition (3.4) as will be illustrated in Example 3A. This is generally not a convenient method for pencil-and-paper calculations. It sometimes may be appropriate for numerical calculations, although there are better methods. (See Note 3.1.) For pencil-and-paper calculations, the Laplace transform method, to be developed in Sec. 3.4, is about as good a method as any.

It should be noted that the state-transition matrix for a time-invariant system is a function only of the *difference* $t - \tau$ between the initial time τ and the present time t as would be expected for a time-invariant system. (See Note 3.2.) Thus, in a time-invariant system, there is no loss in generality in taking the initial time τ to be zero and in computing $\Phi(t) = e^{At}$. If, for a subsequent calculation the initial time is not zero, and $\Phi(t, \tau)$ is needed, it is obtained from $\Phi(t)$ by replacing t by $t - \tau$.

In a time-varying system this procedure is of course not valid; both the initial time and the present time must be treated as general variables. A knowledge of $\Phi(t, 0)$ is not adequate information for the determination of $\Phi(t, \tau)$.

Although the state transition matrix cannot be calculated analytically in general, it is *sometimes* possible to do so because of the very simple structure of the dynamics matrix $A(t)$, as will be illustrated in the missile-guidance example below. Thus, if an application arises in which an expression is necessary for the transition matrix of a time-varying system, the engineer should consider "having a go at it," using whatever ad hoc measures appear appropriate.

Example 3A Motion of mass without friction The differential equation for the position of a mass to which an external force f is applied is

$$\ddot{x} = f/m = u \tag{3A.1}$$

(The control variable $u = f/m$ in this case is the total acceleration.)

Defining the state variables by

$$x_1 = x \qquad x_2 = \dot{x}$$

results in the state-space form

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = u \tag{3A.2}$$

Thus, for this example,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using the series definition (3.4) we obtain the state transition matrix

$$\Phi(t) = e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

The series terminates after only two terms.

The integral in (3.18) with $\tau = 0$ is given by

$$\int_0^t \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} u(\lambda)\,d\lambda = \begin{bmatrix} \int_0^t \lambda u(\lambda)\,d\lambda \\ \int_0^t u(\lambda)\,d\lambda \end{bmatrix}$$

Thus, the solution to (3A.2), using the general formula (3.18) is given by

$$x_1(t) = x_1(0) + tx_2(0) + \int_0^t \lambda u(\lambda)\, d\lambda$$

$$x_2(t) = x_2(0) + \int_0^t u(\lambda)\, d\lambda$$

Obviously these answers could have been obtained directly from (3A.1) without using all the state-space apparatus being developed. This apparatus has its greatest utility when simple methods fail.

Example 3B Missile guidance The equations of motion (assumed to be confined to a plane) of a missile moving at constant speed, relative to a target also moving at constant speed, can be approximated by

$$\dot{\lambda} = \frac{1}{V\bar{T}^2} z \qquad \dot{z} = \bar{T} u \tag{3B.1}$$

where λ is the line-of-sight angle to the target
z is the projected miss distance
V is the velocity of the missile relative to the target
$\bar{T} = T - t$ is the "time-to-go"
u is the acceleration normal to the missile relative velocity vector

It is assumed that the terminal time T is a known quantity. (The reader should review the discussion in Prob. 2.6 for the significance of these variables and the derivation of (3B.1).)

Using the state-variable definitions

$$x_1 = \lambda \qquad x_2 = z$$

results in the matrices

$$A(t) = \begin{bmatrix} 0 & \frac{1}{V\bar{T}^2} \\ 0 & 0 \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 \\ \bar{T} \end{bmatrix} \tag{3B.2}$$

Since $A(t)$ is time-varying (through $\bar{T}$), the transition matrix is *not* the matrix exponential and cannot be found using the series (3.4). In this case, however, we can find the transition matrix by an ad hoc method. First we note that the transition matrix $\Phi(t, \tau)$ expresses the solution to the unforced system

$$\dot{\lambda} = \frac{1}{V\bar{T}^2} z \tag{3B.3}$$

$$\dot{z} = 0 \tag{3B.4}$$

The general form of this solution is

$$\begin{aligned} \lambda(t) &= \phi_{11}(t, \tau)\lambda(\tau) + \phi_{12}(t, \tau)z(\tau) \\ z(t) &= \phi_{21}(t, \tau)\lambda(\tau) + \phi_{22}(t, \tau)z(\tau) \end{aligned} \tag{3B.5}$$

The terms $\phi_{ij}(t, \tau)$ $(i, j) = 1, 2$, which we will now calculate, are the elements of the required transition matrix.

From (3B.4) we have immediately

$$z(t) = z(\tau) = \text{const} \tag{3B.6}$$

Hence

$$\phi_{21}(t, \tau) = 0 \qquad \phi_{22}(t, \tau) = 1 \tag{3B.7}$$

The easiest way to get the first row (ϕ_{11} and ϕ_{12}) of the transition matrix is to use (3B.3) which can be written

$$V(T - \xi)^2 \dot{\lambda}(\xi) = z(\xi) \qquad \text{for all } \xi$$

Thus

$$V(T - \tau)^2 \dot{\lambda}(\tau) = z(\tau)$$

But, from (3B.6), $z(\xi) = z(\tau)$. Hence

$$\dot{\lambda}(\xi) = \frac{1}{(T - \xi)^2} z(\tau) \tag{3B.8}$$

Integrate both sides of (3B.8) from τ to t

$$\int_\tau^t \dot{\lambda}(\xi)\, d\xi = \int_\tau^t \frac{d\xi}{(T - \xi)^2} z(\tau)\, d\xi$$

or

$$\lambda(t) - \lambda(\tau) = \left(\frac{1}{T - t} - \frac{1}{T - \tau}\right) z(\tau) \tag{3B.9}$$

Thus, from (3B.9), we obtain

$$\phi_{11}(t, \tau) = 1 \qquad \phi_{12}(t, \tau) = \frac{1}{T - t} - \frac{1}{T - \tau} \tag{3B.10}$$

Combining (3B.10) with (3B.7) gives the state transition matrix

$$\Phi(t, \tau) = \begin{bmatrix} 1 & \frac{1}{T - t} - \frac{1}{T - \tau} \\ 0 & 1 \end{bmatrix} \tag{3B.11}$$

3.3 INTERPRETATION AND PROPERTIES OF THE STATE-TRANSITION MATRIX

The state-transition matrix, which is fundamental to the theory of linear dynamic systems, has a number of important properties which are the subject of this section.

We note, first of all, that the state-transition matrix is an expression of the solution to the homogeneous equation

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{3.23}$$

where $x(t)$ is given by (3.20). The time derivative of $x(t)$ in (3.20) must of course satisfy (3.23) for any t and $x(t)$. In (3.20) $x(\tau)$ represents initial data and is not a time function. Thus

$$\frac{dx(t)}{dt} = \frac{\partial \Phi(t, \tau)}{\partial t} x(\tau) \tag{3.24}$$

(Since the transition matrix is a function of two arguments t and τ, it is necessary to write its time derivative as a *partial* derivative. The transition matrix also has a derivative with respect to the "initial" time τ which is investigated in Prob. 3.4.) Substitution of (3.24) and (3.20) into (3.23) gives

$$\frac{\partial \Phi(t, \tau)}{\partial t} x(\tau) = A(t)\Phi(t, \tau)x(\tau)$$

Since this must hold for any $x(\tau)$, we may cancel $x(\tau)$ on both sides to finally obtain

$$\frac{\partial \Phi(t, \tau)}{\partial t} = A(t)\Phi(t, \tau) \qquad (3.25)$$

In other words, the transition matrix Φ satisfies the same differential equation as the state x. This can be emphasized by writing (3.25) simply as

$$\dot{\Phi} = A\Phi \qquad (3.26)$$

which does not explicitly exhibit the time dependence of A and Φ. The dot on top of Φ must be interpreted to designate differentiation with respect to the first argument. (Because of the possibility of confusion of arguments use of the full expression (3.25) is recommended in analytical studies.)

We note that (3.20) holds for any t and τ, including $t = \tau$. Thus

$$x(t) = \Phi(t, t)x(t)$$

for any $x(t)$. Thus we conclude that

$$\Phi(t, t) = I \qquad \text{for any } t \qquad (3.27)$$

This becomes the initial condition for (3.25) or (3.26).

Other properties of the transition matrix follow from the fact that the differential equation (3.23) not only possesses a solution for any initial state $x(\tau)$ and any time interval $[\tau, t]$ but that this solution is unique. This is a basic theorem in the theory of ordinary differential equations and is proved in standard textbooks on the subject, e.g., [2, 3]. There are certain restrictions on the nature of permissible time variations of $A(t)$ but these are always satisfied in real-world systems. When A is a constant matrix, of course, not only do we know that Φ exists but we have an expression for it, namely $\Phi(t) = e^{At}$.

Assuming the existence and uniqueness of solutions, we can write

$$x(t_3) = \Phi(t_3, t_1)x(t_1) \qquad \text{for any } t_3, t_1 \qquad (3.28)$$

and also

$$x(t_3) = \Phi(t_3, t_2)x(t_2) \qquad \text{for any } t_3, t_2 \qquad (3.29)$$

$$x(t_2) = \Phi(t_2, t_1)x(t_1) \qquad \text{for any } t_2, t_1 \qquad (3.30)$$

Thus, substituting (3.30) into (3.29)

$$x(t_3) = \Phi(t_3, t_2)\Phi(t_2, t_1)x(t_1) \qquad (3.31)$$

Comparing (3.31) with (3.28) we see that

$$\Phi(t_3, t_1) = \Phi(t_3, t_2)\Phi(t_2, t_1) \qquad \text{for any } t_3, t_2, t_1 \qquad (3.32)$$

This very important property—known as the *semigroup* property—of the state-transition matrix is a direct consequence of the fact that whether we go from state $x(t_1)$ to $x(t_3)$ directly or via an "intermediate" state $x(t_2)$, we must end at the same point. Note, however, that the time t_2 of the intermediate state need not be between t_1 and t_3.

The semigroup properties (3.32) and (3.27) gives

$$I = \Phi(t, \tau)\Phi(\tau, t)$$

or

$$\Phi(\tau, t) = [\Phi(t, \tau)]^{-1} \qquad \text{for any } t, \tau \qquad (3.33)$$

This of course means that the state-transition matrix is never singular even if the dynamics matrix A is singular, as it often is.

In a time-invariant system, the transition matrix is characterized by a single argument, as already discussed:

$$\Phi(t_1, t_2) = \Phi(t_1 - t_2)$$

Thus, for time-invariant systems, the properties (3.27), (3.32), and (3.33) become

$$\Phi(0) = I \qquad (3.34)$$

$$\Phi(t)\Phi(\tau) = \Phi(t + \tau) \qquad (3.35)$$

$$\Phi^{-1}(t) = \Phi(-t) \qquad (3.36)$$

It is readily verified that $\Phi(t) = e^{At}$ possesses these properties:

$$e^{A0} = I \qquad (3.37)$$

$$e^{At}e^{A\tau} = e^{A(t+\tau)} \qquad (3.38)$$

$$(e^{At})^{-1} = e^{-At} \qquad (3.39)$$

The first relation (3.37) is apparent from the series definition (3.4) and the second relation (3.38) can be verified by multiplying the series for e^{At} by the series for $e^{A\tau}$. (The calculations are a bit tedious, but the skeptical reader is invited to perform them.) The third relation (3.39) follows from the first two.

By analogy with (3.38) the reader might be tempted to conclude that $e^{At}e^{Bt} = e^{(A+B)t}$. This is generally *not* true, however. In order for it to be true A and B must commute (i.e., $AB = BA$) and this condition is rarely met in practice.

3.4 SOLUTION BY THE LAPLACE TRANSFORM: THE RESOLVENT

As the reader is no doubt aware, Laplace transforms are very useful for solving time-invariant differential equations. Indeed Laplace transforms are the basis of the entire frequency-domain methodology, to which the next chapter is devoted.

The Laplace transform of a signal $f(t)$ which may be an input variable or a state variable is defined by

$$\mathcal{L}[f(t)] = \mathsf{f}(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad (3.40)$$

where s is a complex variable generally called *complex frequency*. A discussion of the region of convergence of $\mathsf{f}(s)$ in the complex s plane, and many other details about the Laplace transform are to be found in many standard textbooks such as [1] and [4].

The sans-serif letter f used to designate the Laplace transform of $f(t)$ was chosen advisedly. In texts in which the signals are all scalars, capital letters are used to denote Laplace transforms (viz., $X(s) = \mathcal{L}[x(t)]$, $Y(s) = \mathcal{L}[y(t)]$, etc.). But in this book capital letters have been preempted for designating matrices. The use of sans-serif letters for Laplace transforms avoids the risk of confusion.

The lower limit on the integral has been written as 0. In accordance with engineering usage, this is understood to be 0^-, that is, the instant just prior to the occurrence of discontinuities, impulses, etc., in the signals under examination. The reader who is unfamiliar with this usage should consult a standard text such as [1] or [4].

The Laplace transform is useful for solving (3.1) only when A and B are constant matrices which we will henceforth assume. In order to use the Laplace transform, we need an expression for the Laplace transform of the time derivative of $f(t)$

$$\mathcal{L}[\dot{f}(t)] = \int_0^\infty e^{-st}\frac{df}{dt}\,dt = e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -s\,e^{-st}f(t)\,dt \qquad (3.41)$$

upon integration by parts. Assuming

$$\lim_{t\to\infty} e^{-st}f(t) \to 0$$

(3.41) becomes

$$\mathcal{L}[\dot{f}(t)] = s\int_0^\infty e^{-st}f(t)\,dt - f(0) = s\mathsf{f}(s) - f(0) \qquad (3.42)$$

We also note that (3.42) applies when $f(t)$ is a vector:

$$\mathcal{L}[f(t)] = \mathcal{L}\begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t)\end{bmatrix} = \begin{bmatrix} \mathcal{L}[f_1(t)] \\ \vdots \\ \mathcal{L}[f_n(t)]\end{bmatrix} = \begin{bmatrix} \mathsf{f}_1(s) \\ \vdots \\ \mathsf{f}_n(s)\end{bmatrix} = \mathsf{f}(s) \qquad (3.43)$$

and also that

$$\mathcal{L}[Ax(t)] = A\mathsf{x}(s) \qquad (3.44)$$

Applying all of these to (3.1) with A and B *constant* gives

$$s\mathsf{x}(s) - x(0) = A\mathsf{x}(s) + B\mathsf{u}(s)$$

or,

$$(sI - A)\mathsf{x}(s) = x(0) + B\mathsf{u}(s)$$

Solve for $\mathsf{x}(s)$ to obtain

$$\mathsf{x}(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\mathsf{u}(s) \qquad (3.45)$$

On taking the inverse Laplace transform of $\mathsf{x}(s)$ as given by (3.45) we obtain the desired solution for $x(t)$. We note that $\mathsf{x}(s)$ is the sum of two terms, the first due to the initial condition $x(0)$ multiplied by the matrix $(sI - A)^{-1}$ and the second being the product of this matrix and the term due to the input $B\mathsf{u}(s)$. Knowing the inverse Laplace transform of $(sI - A)^{-1}$ would permit us to find the inverse Laplace transform of (3.45) and hence obtain $x(t)$. In the scalar case we recall that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} = (s-a)^{-1} \qquad (3.46)$$

We have not yet discussed calculating the Laplace transform of a matrix function of time. But we should not be very much surprised to learn that

$$\mathcal{L}[e^{At}] = (sI - A)^{-1} \qquad (3.47)$$

which is simply the matrix version of (3.46). It can be shown by direct calculation (see Note 3.3) that (3.47) is in fact true. And if this be the case then the inverse Laplace transform of (3.45) is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \qquad (3.48)$$

which is the desired solution. The integral term in (3.48) is given by the well-known *convolution theorem* for the Laplace transform [1]

$$\mathcal{L}\left[\int_0^t f(t-\lambda)g(\lambda)\,d\lambda\right] = \mathsf{f}(s)\mathsf{g}(s)$$

which is readily extended from scalar functions to matrices.

The solution for $x(t)$ given by (3.48) is a special case (namely $\tau = 0$) of the general solution (3.16) obtained by another method of analysis. This confirms, if confirmation is necessary, the validity of (3.47).

The exponential matrix e^{At} is known as the state transition matrix (for a time invariant system) and its Laplace transform

$$\Phi(s) = (sI - A)^{-1} \qquad (3.49)$$

is known in mathematical literature as the *resolvent* of A. In engineering literature this matrix has been called the *characteristic frequency* matrix[1] or simply the *characteristic matrix*.[4] Regrettably there doesn't appear to be a standard symbol for the resolvent, which we have designated as $\Phi(s)$ in this book.

The fact that the state transition matrix is the inverse Laplace transform of the resolvent matrix facilitates the calculation of the former. It also characterizes the dynamic behavior of the system, the subject of the next chapter. The steps one takes in calculating the state-transition matrix using the resolvent are:

(*a*) Calculate $sI - A$.
(*b*) Obtain the resolvent by inverting $(sI - A)$.
(*c*) Obtain the state-transition matrix by taking the inverse Laplace transform of the resolvent, element by element.

The following examples illustrate the process.

Example 3C DC motor with inertial load In Chap. 2 (Example 2B) we found that the dynamics of a dc motor driving an inertial load are

$$\dot\theta = \omega$$
$$\dot\omega = -\alpha\omega + \beta u$$

The matrices of the state-space characterization are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

Thus the resolvent is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+\alpha \end{bmatrix}^{-1} = \frac{1}{s(s+\alpha)}\begin{bmatrix} s+\alpha & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+\alpha)} \\ 0 & \dfrac{1}{s+\alpha} \end{bmatrix}$$

Finally, taking the inverse Laplace transforms of each term in $\Phi(s)$ we obtain

$$e^{At} = \Phi(t) = \begin{bmatrix} 1 & (1 - e^{-\alpha t})/\alpha \\ 0 & e^{-\alpha t} \end{bmatrix}$$

Example 3D Inverted pendulum The equations of motion of an inverted pendulum were determined to be (approximately)

$$\dot\theta = \omega$$
$$\dot\omega = \Omega^2\theta + u$$

Hence the matrices of the state-space characterization are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The resolvent is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -\Omega^2 & s \end{bmatrix}^{-1} = \frac{1}{s^2 - \Omega^2}\begin{bmatrix} s & 1 \\ \Omega^2 & s \end{bmatrix}$$

and the state-transition matrix is

$$\Phi(t) = e^{At} = \begin{bmatrix} \cosh\Omega t & \sinh\Omega t/\Omega \\ \Omega\sinh\Omega t & \cosh\Omega t \end{bmatrix}$$

For a general kth-order system the matrix $sI - A$ has the following appearance

$$sI - A = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1k} \\ -a_{21} & s - a_{22} & \cdots & -a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & -a_{k2} & \cdots & s - a_{kk} \end{bmatrix} \tag{3.50}$$

We recall (see Appendix) that the inverse of any matrix M can be written as the adjoint matrix, adj M, divided by the determinant $|M|$. Thus

$$(sI - A)^{-1} = \frac{\operatorname{adj}(sI - A)}{|sI - A|}$$

If we imagine calculating the determinant $|sI - A|$ we see that one of the terms will be the product of the diagonal elements of $sI - A$:

$$(s - a_{11})(s - a_{22})\cdots(s - a_{kk}) = s^k + c_1 s^{k-1} + \cdots + c_k$$

a polynomial of degree k with the leading coefficient of unity. There will also be other terms coming from the off-diagonal elements of $sI - A$ but none will have a degree as high as k. Thus we conclude that

$$|sI - A| = s^k + a_1 s^{k-1} + \cdots + a_k \tag{3.51}$$

This is known as the *characteristic polynomial* of the matrix A. It plays a vital role in the dynamic behavior of the system. The roots of this polynomial are called the *characteristic roots*, or the *eigenvalues*, or the *poles*, of the system and determine the essential features of the unforced dynamic behavior of the system, since they determine the inverse Laplace transform of the resolvent, which is the transition matrix. See Chap. 4.

The adjoint of a k by k matrix is itself a k by k matrix whose elements are the cofactors of the original matrix. Each cofactor is obtained by computing the determinant of the matrix that remains when a row and a column of the original matrix are deleted. It thus follows that each element in adj$(sI - A)$ is a polynomial in s of maximum degree $k - 1$. (The polynomial cannot have degree k when any row and column of $sI - A$ is deleted.) Thus it is seen that the adjoint of $sI - A$ can be written

$$\operatorname{adj}(sI - A) = E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k$$

Thus we can express the resolvent in the following form

$$(sI - A)^{-1} = \frac{E_1 s^{k-1} + \cdots + E_k}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{3.52}$$

An interesting and useful relationship for the coefficient matrices E_i of the adjoint matrix can be obtained by multiplying both sides of (3.52) by $|sI - A|(sI - A)$. The result is

$$|sI - A|I = (sI - A)(E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k) \tag{3.53}$$

or

$$s^k I + a_1 s^{k-1} I + \cdots + a_k I = s^k E_1 + s^{k-1}(E_2 - AE_1) + \cdots + s(E_k - AE_{k-1}) - AE_k$$

Equating the coefficients of s^i on both sides of (3.53) gives

$$\begin{aligned} E_1 &= I \\ E_2 - AE_1 &= a_1 I \\ E_3 - AE_2 &= a_2 I \\ &\cdots\cdots \\ E_k - AE_{k-1} &= a_{k-1} I \\ -AE_k &= a_k I \end{aligned} \tag{3.54}$$

We have thus determined that the leading coefficient matrix of adj $(sI - A)$ is the identity matrix, and that the subsequent coefficients can be obtained recursively:

$$\begin{aligned} E_2 &= AE_1 + a_1 I \\ E_3 &= AE_2 + a_2 I \\ &\cdots\cdots \\ E_k &= AE_{k-1} + a_{k-1} I \end{aligned} \tag{3.55}$$

The last equation in (3.54) is redundant, but can be used as a check, when the recursion equations (3.55) are used as the basis of a numerical algorithm. In this case the "check equation" can be written

$$E_{k+1} = AE_k + a_k I = 0 \tag{3.56}$$

An algorithm based on (3.55) requires the coefficients a_i $(i = 1, \ldots, k)$ of the characteristic polynomial. Fortunately, the determination of these coefficients can be included in the algorithm, for it can be shown that

$$a_1 = -\mathrm{tr}\,(AE_1)$$

$$a_2 = -\tfrac{1}{2}\,\mathrm{tr}\,(AE_2)$$

More generally

$$a_i = -\frac{1}{i}\,\mathrm{tr}\,(AE_i) \qquad i = 1, 2, \ldots, k \tag{3.57}$$

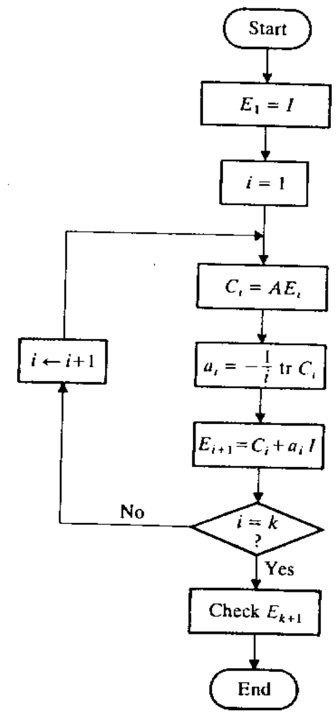

Figure 3.1 Algorithm for computing

$$(sI - A)^{-1} = \frac{E_1 s^{k-1} + \cdots + E_k}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

An algorithm for computing the numerator matrices E_i and the coefficients a_i, starting with $E_1 = I$, is illustrated in the form of a flow chart in Fig. 3.1.

A proof of (3.57) is found in many textbooks such as [5, 6]. The algorithm based on (3.56) and (3.57) appears to have been discovered several times in various parts of the world. The names of Leverrier, Souriau, Faddeeva, and Frame are often associated with it.

This algorithm is convenient for hand calculation and easy to implement on a digital computer. Unfortunately, however, it is not a very good algorithm when the order k of the system is large (higher than about 10). The check matrix E_{k+1}, which is supposed to be zero, usually turns out to be embarrassingly large, and hence the resulting coefficients a_i and E_i are often suspect.

Example 3E Inertial navigation The equations for errors in an inertial navigation system are approximated by

$$\begin{aligned} \Delta\dot{x} &= \Delta v \\ \Delta\dot{v} &= -g\Delta\psi + E_A \\ \Delta\dot{\psi} &= \frac{1}{R}\Delta v + E_G \end{aligned} \tag{3E.1}$$

where Δx is the position error, Δv is the velocity error, $\Delta\psi$ is the tilt of the platform, g is the acceleration of gravity, and R is the radius of the earth. (The driving terms are the accelerometer error E_A and the gyro error E_G.)

For the state variables defined by

$$x_1 = \Delta x \qquad x_2 = \Delta v \qquad x_3 = \Delta\psi$$

the A matrix is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix}$$

and, regarding E_A and E_G as inputs, the B matrix is

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrices appearing in the recursive algorithm are

$$C_1 = AE_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix} \qquad a_1 = -\text{tr}\, C_1 = 0 \qquad E_2 = C_1 + a_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix}$$

$$C_2 = AE_2 = \begin{bmatrix} 0 & 0 & -g \\ 0 & -g/R & 0 \\ 0 & 0 & -g/R \end{bmatrix} \qquad a_2 = -\tfrac{1}{2}(-2g/R) = g/R \qquad E_3 = C_2 + a_2 I = \begin{bmatrix} g/R & 0 & -g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 = AE_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad a_3 = 0 \qquad E_4 = C_3 + a_3 I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$(sI - A)^{-1} = \begin{bmatrix} s^2 + g/R & s & -g \\ 0 & s^2 & -gs \\ 0 & s/R & s^2 \end{bmatrix} \frac{1}{s^3 + (g/R)s}$$

$$= \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2 + g/R} & \frac{-g}{s(s^2 + g/R)} \\ 0 & \frac{s}{s^2 + g/R} & \frac{-g}{s^2 + g/R} \\ 0 & \frac{1/R}{s^2 + g/R} & \frac{s}{s^2 + g/R} \end{bmatrix} \tag{3E.2}$$

The state transition matrix corresponding to the resolvent (3E.2) is obtained by taking its inverse Laplace transform.

$$\Phi(t) = \begin{bmatrix} 1 & \frac{\sin \Omega t}{\Omega} & \frac{g}{\Omega^2}(\cos \Omega t - 1) \\ 0 & \cos \Omega t & -\frac{g}{\Omega}\sin \Omega t \\ 0 & \frac{\sin \Omega t}{\Omega R} & \cos t \end{bmatrix} \qquad \Omega = \sqrt{g/R} \tag{3E.3}$$

The elements of the state transition matrix, with the exception of ϕ_{11} are all oscillatory with a frequency $\Omega = \sqrt{g/R}$ which is the natural frequency of a pendulum of length equal to the earth's radius; $\Omega = 0.001\,235$ rad/s corresponding to a period $T = 2\pi/\Omega = 84.4$ min., which is known as the "Schuler period." (See Note 3.4.)

Because the error equations are undamped, the effects of even small instrument biases can result in substantial navigation errors. Consider, for example, a constant gyro bias

$$E_G = \frac{c}{s}$$

The Laplace transform of the position error is given by

$$\Delta x(s) = \phi_{13}(s)\frac{c}{s} = -\frac{g}{s^2(s^2 + \Omega^2)}c \tag{3E.4}$$

and the corresponding position error, as a function of time, is the inverse Laplace transform of (3E.4)

$$\Delta x(t) = -\frac{g}{\Omega^2}\left(t - \frac{1}{\Omega}\sin \Omega t\right)c \tag{3E.5}$$

The position error consists of two terms: a periodic term at the Schuler period and a term which grows with time (also called a *secular term* at a rate of $-(g/\Omega^2)c = -Rc$. The position error thus grows at a rate proportional to the earth's radius. The position error will grow at a rate of about 70 m/h for each degree-per-hour "drift" ($E_G = c$) of the gyro.

3.5 INPUT-OUTPUT RELATIONS: TRANSFER FUNCTIONS

In conventional (frequency-domain) analysis of system dynamics attention is focused on the relationship between the output y and the input u. The focus shifts to the state vector when state space analysis is used, but there is still an interest in the input-output relation. Usually when an input-output analysis is made, the initial state $x(0)$ is assumed to be zero. In this case the Laplace transform of the state is given by

$$\mathrm{x}(s) = (sI - A)^{-1}B\mathrm{u}(s) \tag{3.58}$$

If the output is defined by

$$y(t) = Cx(t) \tag{3.59}$$

Then its Laplace transform is

$$\mathrm{y}(s) = C\mathrm{x}(s) \tag{3.60}$$

and, by (3.58)

$$\mathrm{y}(s) = C(sI - A)^{-1}B\mathrm{u}(s) \tag{3.61}$$

The matrix

$$\mathrm{H}(s) = C(sI - A)^{-1}B \tag{3.62}$$

that relates the Laplace transform of the output to the Laplace transform of the input is known as the *transfer-function* matrix.

The inverse Laplace transform of the transfer-function matrix

$$H(t) = \mathcal{L}^{-1}[\mathrm{H}(s)] = Ce^{At}B \tag{3.63}$$

is known as the *impulse-response* matrix. In the time domain $y(t)$ can be expressed by the convolution of the impulse-response matrix with the input

$$y(t) = \int_0^t H(t-\lambda)u(\lambda)\,d\lambda = \int_0^t Ce^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.64}$$

This relationship is equivalent to (3.48) in which the initial state $x(0)$ is assumed to be zero, with (3.59) relating $y(t)$ to $x(t)$.

If there is a direct path from the input to the output owing to the presence of a matrix D

$$y(t) = Cx(t) + Du(t)$$

Then

$$\mathrm{y}(s) = C\mathrm{x}(s) + D\mathrm{u}(s)$$

and the transfer-function matrix

$$\mathrm{H}(s) = C(sI - A)^{-1}B + D \tag{3.65}$$

with the corresponding impulse-response matrix

$$H(t) = Ce^{At}B + D\delta(t) \tag{3.66}$$

The delta function (unit impulse) appears in (3.66) because of the direct connection, through D, from the input to the output. Since the impulse response of a system is defined as the output $y(t)$ when the input $u(t) = \delta(t)$, it is clear that the output must contain $D\delta(t)$. If the direct connection from the input to the output is absent, the impulse response does not contain an impulse term. This implies that the degree of the numerator in $\mathrm{H}(s)$ must be lower than the degree of the denominator. Since the adjoint matrix of $sI - A$ is of the degree $k-1$ (see (3.52)) then the degree of $\mathrm{H}(s)$ is no higher than $k-1$. Specifically, with $D = 0$

$$\mathrm{H}(s) = \frac{C[E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k]B}{|sI - A|}$$
$$= \frac{CBs^{k-1} + CE_2Bs^{k-2} + \cdots + CE_kB}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{3.67}$$

Thus the transfer-function matrix is a rational function of s with the numerator of degree $k-1$ (or less) and the denominator of degree k.

Example 3F Missile dynamics Except for difference in size, weight, and speed a missile is simply a pilotless aircraft. Hence the aerodynamic equations of a missile are the same as those of an aircraft, namely (2.40) and (2.41).

In many cases the coupling of the change of velocity Δu normal to the longitudinal axis into the equations for angle of attack α and pitch rate q is negligible: Z_u, M_u, X_u are

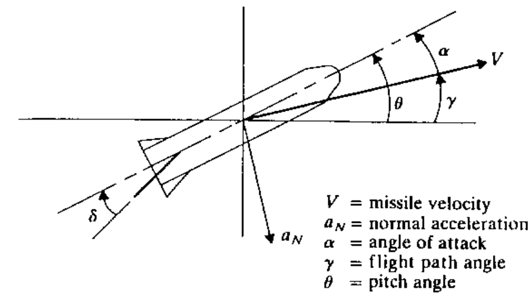

Figure 3.2 Missile dynamic variables.

insignificant. In this case (2.40) gives the following pitch dynamics:

$$\dot{\alpha} = \frac{Z_\alpha}{V}\alpha + q + \frac{Z_\delta}{V}\delta$$
$$\dot{q} = M_\alpha \alpha + M_q q + M_\delta \delta \tag{3F.1}$$

where δ is the control surface deflection. (The control surface may be located in front of the missile—in which case it is called a *canard*—or in the more familiar aft position. Its location with respect to the center of mass of the missile will determine the signs of the Z_δ and M_δ, used here instead of Z_E and M_E which were introduced in Chap. 2.)

The pitch angle θ is usually not of interest, hence the differential equation $\dot{\theta} = q$ can be omitted.

Missile guidance laws are generally expressed in terms of the component of acceleration normal to the velocity vector of the missile; in proportional navigation, for example, it is desired that this acceleration be proportional to the inertial line-of-sight rate. (See Example 9G.) Thus the output of interest in a typical missile is the "normal" component of acceleration a_N. In the planar case (see Fig. 3.2)

$$a_N \approx -V\dot{\gamma} \tag{3F.2}$$

where γ is the flight path angle. But

$$\gamma = \theta - \alpha$$

or

$$\dot{\gamma} = q - \dot{\alpha} \tag{3F.3}$$

Thus, using (3F.2) and (3F.1),

$$a_N \approx Z_\alpha \alpha + Z_\delta \delta \tag{3F.4}$$

With the state, input, and output of the missile defined respectively by

$$x = \begin{bmatrix} \alpha \\ q \end{bmatrix} \qquad u = \delta \qquad y = a_N$$

the matrices of the standard representation $\dot{x} = Ax + Bu$, $y = Cx + Du$ are

$$A = \begin{bmatrix} Z_\alpha/V & 1 \\ M_\alpha & M_q \end{bmatrix} \qquad B = \begin{bmatrix} Z_\delta/V \\ M_\delta \end{bmatrix}$$

$$C = [Z_\alpha \quad 0] \qquad D = [Z_\delta]$$

A block-diagram representation of the system is shown in Fig. 3.3.

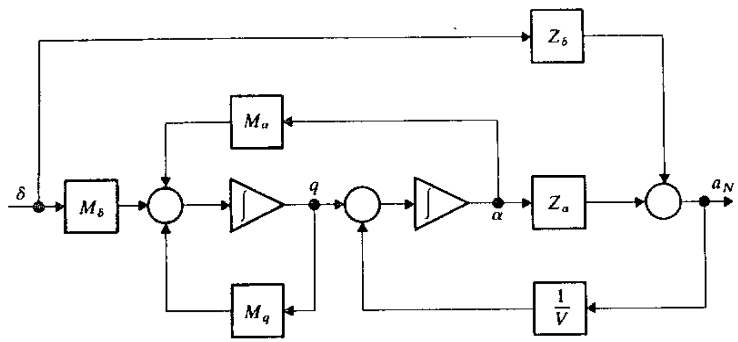

Figure 3.3 Block-diagram of missile dynamics showing normal acceleration as output.

The transfer function from the input $u = \delta$ to the output $y = a_N$ is given by

$$H(s) = C(sI - A)^{-1}B + D$$

$$= [Z_\alpha \quad 0]\begin{bmatrix} s - Z_\alpha/V & -1 \\ -M_\alpha & s - M_q \end{bmatrix}^{-1}\begin{bmatrix} Z_\delta/V \\ M_\delta \end{bmatrix} + Z_\delta$$

$$= \frac{Z_\delta(s^2 - M_q s - M_\alpha) + Z_\alpha M_\delta}{s^2 - \left(M_q + \frac{Z_\alpha}{V}\right)s + \frac{Z_\alpha}{V}M_q - M_\alpha} \qquad (3F.5)$$

In a typical missile Z_α, M_α, Z_δ, and M_δ are all negative. Thus the coefficient of s^2 in the numerator of H(s) in (3F.5) is negative. The constant term $Z_\alpha M_\delta - M_\alpha Z_\delta$, on the other hand, is typically positive. This implies that the numerator of H(s) has a zero in the right half of the s plane. A transfer function having a right-half plane zero is said to be "nonminimum-phase" and can be the source of considerable difficulty in design of a well-behaved closed-loop control system. One can imagine the problem that might arise by observing that the dc gain $-(Z_\alpha M_\delta - M_\alpha Z_\delta)/M_\alpha$ is (typically) positive but the high-frequency gain $-Z_\delta/M_\alpha$ is (typically) negative. So if a control law is designed to provide negative feedback at dc, unless great care is exercised in the design, it is liable to produce positive feedback at high frequencies. Another peculiarity of the transfer function of (3F.5) is that its step response starts out

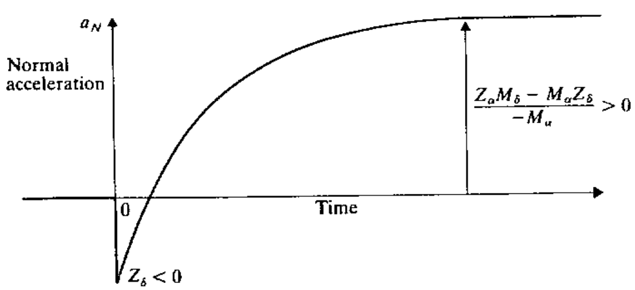

Figure 3.4 Normal acceleration step response (open-loop) of tactical missile showing reversal in sign.

negative and then turns positive, as shown in Fig. 3.4. The initial value of the step response is

$$\lim_{s\to\infty} s\left[\frac{1}{s}H(s)\right] = Z_\delta < 0 \qquad \text{(typically)}$$

but the final value of the step response is

$$\lim_{s\to 0} s\left[\frac{1}{s}H(s)\right] = \frac{Z_\alpha M_\delta - M_\alpha Z_\delta}{-M_\alpha} > 0 \qquad \text{(typically)}$$

Example 3G Dynamics of two-axis gyroscope In Example 2F we used the general theory of rigid-body dynamics, and made small angle approximations to develop the equations of motion for a two-axis gyroscope ("gyro"):

$$\dot{\delta}_x = \omega_{xB} - \omega_{xE}$$

$$\dot{\delta}_y = \omega_{yB} - \omega_{yE}$$

$$\dot{\omega}_{xB} = \frac{H}{J_d}\omega_{yB} - \frac{B}{J_d}(\omega_{xB} - \omega_{xE}) - \frac{K_D}{J_d}\delta_x - \frac{K_Q}{J_d}\delta_y + \frac{\tau_x}{J_d} \qquad (3G.1)$$

$$\dot{\omega}_{yB} = -\frac{H}{J_d}\omega_{xB} - \frac{B}{J_d}(\omega_{yB} - \omega_{yE}) + \frac{K_Q}{J_d}\delta_x - \frac{K_D}{J_d}\delta_y + \frac{\tau_y}{J_d}$$

where δ_x and δ_y are the angular displacements of the gyro rotor about x and y axes with respect to the case; ω_{xB} and ω_{yB} are the components of the inertial velocity of the rotor projected onto the x and y axes of the gyro; ω_{xE}, ω_{yE} are the angular velocity components of the gyro case projected onto the same axes; τ_x and τ_y are the externally supplied control torques. The parameters H, J_D, K_D, K_Q are physical parameters of the gyro, as explained in Example 2F.

With respect to the dynamic model of (3G.1), there are two kinds of inputs: *control* inputs, represented by the control torques τ_x and τ_y, and *exogenous* inputs, represented by the case angular-velocity components ω_x and ω_y. These exogenous inputs are not "disturbances" in the sense of being unwanted; their presence is the raison d'être for the gyro.

The standard vector matrix form of (3G.1) is thus

$$\dot{x} = Ax + Bu + Ex_0$$

where

$$x = \begin{bmatrix} \delta_x \\ \delta_y \\ \omega_{xB} \\ \omega_{yB} \end{bmatrix} \qquad u = \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} \qquad x_0 = \begin{bmatrix} \omega_{xE} \\ \omega_{yE} \end{bmatrix}$$

$$A = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -K_D/J_d & -K_Q/J_d & -B/J_d & H/J_d \\ K_Q/J_d & -K_D/J_d & -H/J_d & -B/J_d \end{array}\right] \qquad (3G.2)$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/J_d & 0 \\ 0 & 1/J_d \end{bmatrix} \qquad E = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ B/J_d & 0 \\ 0 & B/J_d \end{bmatrix} \qquad (3G.3)$$

The special structure of the lower half of the A matrix is noteworthy: The 2×2 submatrix in the lower right-hand corner is

$$\begin{bmatrix} -B/J_d & -H/J_d \\ H/J_d & -B/J_d \end{bmatrix} = -\frac{B}{J_d}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{H}{J_d}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (3G.4)$$

The B/J_d terms are conventional damping terms (torque proportional to angular velocity) which tend to dissipate the initial energy of the gyro. The H/J_d terms (which appear in a skew symmetric matrix) have an entirely different effect: They do not cause the energy of the gyro to dissipate but rather produce a high-frequency oscillation called "nutation," a phenomenon present in all gyros, to be discussed at greater length later.

The 2×2 submatrix in the lower left-hand corner of the A matrix is also of interest. This matrix is

$$\begin{bmatrix} -K_D/J_d & -K_Q/J_d \\ K_Q/J_d & -K_D/J_d \end{bmatrix} = -\frac{K_D}{J_d}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{K_Q}{J_d}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3G.5}$$

The K_D/J_d terms are conventional spring terms. In a gyro they give rise to a low-frequency oscillatory motion known as "precession."

We can evince these phenomena by studying the characteristic equation of the gyro:

$$|sI - A| = \begin{vmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ c_1 & c_2 & s + b_1 & b_2 \\ -c_2 & c_1 & -b_2 & s + b_1 \end{vmatrix} = 0 \tag{3G.6}$$

where

$$b_1 = B/J_d \qquad b_2 = H/J_d$$
$$c_1 = K_D/J_d \qquad c_2 = K_Q/J_d$$

The determinant appearing in (3G.6) can be evaluated in a variety of ways—some simpler than others. The result is

$$|sI - A| = (s^2 + b_1 s + c_1)^2 + (b_2 s + c_2)^2 = 0 \tag{3G.7}$$

or

$$(s^2 + b_1 s + c_1)^2 = -(b_2 s + c_2)^2$$

Thus the eigenvalues are the roots of

$$s^2 + b_1 s + c_1 = \pm j(b_2 s + c_2)$$

or

$$s^2 + (b_1 \mp jb_2)s + c_1 \mp jc_2 = 0 \tag{3G.8}$$

The eigenvalues of the system are thus the four roots of (3G.8)

$$s = \frac{-(b_1 \mp jb_2) \pm \sqrt{(b_1 \mp jb_2)^2 - 4(c_1 \mp jc_2)}}{2} \tag{3G.9}$$

In an ideal gyro the "spring" coefficients c_1 and c_2 are zero; they are not zero in some types of real gyros, but in any case they are very small; i.e.,

$$|c_1 + jc_2| \ll |b_1 + jb_2|^2 \tag{3G.10}$$

Taking note of this, we write the radical in (3G.9) as

$$\sqrt{(b_1 \mp jb_2)^2 - 4(c_1 \mp jc_2)} = (b_1 \mp jb_2)\sqrt{1 - \frac{4(c_1 \mp jc_2)}{(b_1 \mp jb_2)^2}} \tag{3G.11}$$

Using the approximation:

$$(1 + \varepsilon)^{1/2} \approx 1 + \tfrac{1}{2}\varepsilon \qquad \text{for } \varepsilon \ll 1$$

we obtain for (3G.11)

$$\sqrt{(b_1 \mp jb_2)^2 - 4(c_1 \mp jc_2)} \approx b_1 \mp jb_2 - \frac{2(c_1 \mp jc_2)}{b_1 \mp jb_2}$$

Now

$$\frac{c_1 \mp jc_2}{b_1 \mp jb_2} = \frac{c_1 \mp jc_2}{b_1 \mp jb_2}\,\frac{b_1 \pm jb_2}{b_1 \pm jb_2} = \frac{(b_1c_1 + b_2c_2) \mp j(b_1c_2 - b_2c_1)}{b_1^2 + b_2^2}$$

Hence, by (3G.9), the approximate poles are given by

$$s = -b_1 + \frac{b_1c_1 + b_2c_2}{b_1^2 + b_2^2} \pm j\left(b_2 + \frac{b_2c_1 - b_1c_2}{b_1^2 + b_2^2}\right) \tag{3G.12}$$

and

$$s = -\frac{b_1c_1 + b_2c_2}{b_1^2 + b_2^2} \pm j\frac{b_2c_1 - b_1c_2}{b_1^2 + b_2^2} \tag{3G.13}$$

On the complex plane, the four eigenvalues are positioned as shown in Fig. 3.5. Two eigenvalues are located relatively close to the origin at a natural frequency

$$\omega_p = \frac{b_2c_1 - b_1c_2}{b_1^2 + b_2^2} \tag{3G.14}$$

which is known as the *precession* frequency. The pole stable with a (negative) real part

$$\alpha_p = -\frac{b_1c_1 + b_2c_2}{b_1^2 + b_2^2} \tag{3G.15}$$

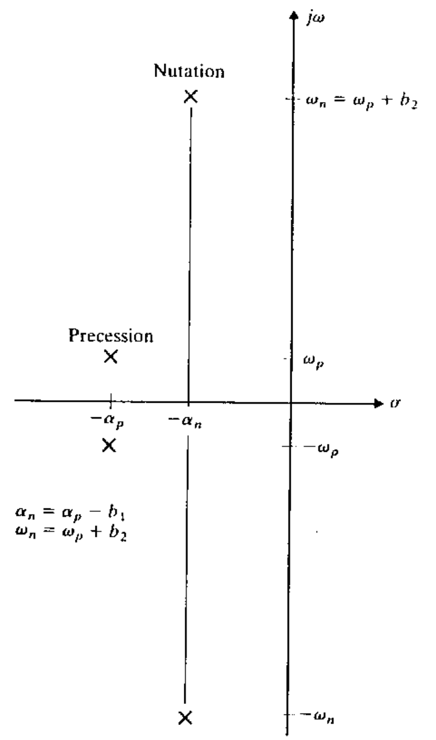

Figure 3.5 Poles of two-axis gyroscope.

The other two poles are located much farther from the origin, at a natural frequency

$$\omega_n = b_2 + \omega_p$$

which is known as the "nutation" frequency. This pole is also stable with a (negative) real part of

$$\alpha_n = -b_1 + \alpha_p \tag{3G.16}$$

The precession poles are due to the presence of the spring terms c_1 and c_2. In an ideal gyro in which these terms are absent, the precession poles move to the origin and the nutation terms become

$$\begin{aligned} \omega_n &= b_2 = H/J_d \\ \alpha_n &= -b_1 = B/J_d \end{aligned} \tag{3G.17}$$

With the precession terms present, the nutation frequency changes from H/J_d by the amount of the precession frequency, and the damping is decreased.

The outputs of the gyro are the signal measured at the pick-off angles. Thus the output equations are

$$y_1 = \delta_x$$
$$y_2 = \delta_y$$

or, in vector-matrix notation

$$y = Cx$$

with

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The transfer-function matrix from the external inputs ω_x and ω_y to the observed outputs δ_x and δ_y is

$$\mathrm{H}_l(s) = C(sI - A)^{-1}E \tag{3G.18}$$

and the transfer-function matrix from the control inputs τ_x and τ_y to the output is

$$\mathrm{H}_u(s) = C(sI - A)^{-1}B \tag{3G.19}$$

On evaluating (3G.18) we find the matrix of transfer functions for the *free* (uncontrolled) gyro

$$\mathrm{H}_l(s) = \frac{\begin{bmatrix} s^2 + b_1 s + c_1 & -b_2 s - c_2 \\ b_2 s + c_2 & s^2 + b_1 s + c_1 \end{bmatrix}}{(s^2 + b_1 s + c_1)^2 - (b_2 s + c_2)^2}$$

For inertial navigation purposes, an ideal gyro is one in which all the parameters are zero with the exception of $b_2 = H/J$. In this ideal case

$$\mathrm{H}_l(s) = \frac{\begin{bmatrix} s^2 & -b_2 s \\ b_2 s & s^2 \end{bmatrix}}{s^2(s^2 + b_2^2)}$$

For a step input of angular velocity, say

$$\Omega_x(s) = 1/s \qquad \Omega_y(s) = 0$$

the Laplace transforms of the outputs are

$$\Delta_x(s) = \frac{1}{s(s^2 + b_2^2)} = \frac{1}{b_2^2}\left(\frac{1}{s} - \frac{s}{s^2 + b_2^2}\right)$$

$$\Delta_y(s) = -\frac{1}{s^2(s^2 + b_2^2)} = -\frac{1}{b_2}\left(\frac{1}{s^2} - \frac{1}{s^2 + b_2^2}\right)$$

and the corresponding time functions are

$$\delta_x(t) = \frac{1}{b_2^2}(1 - \cos b_2 t)$$

$$\delta_y(t) = -\frac{1}{b_2}\left(t - \frac{1}{b_2}\sin b_2 t\right)$$

as shown in Fig. 3.6. The output angle δ_x for an angular velocity input about the x axis is a sinusoid of amplitude $1/b_2^2$ with a dc value of $1/b_2^2$. The cross-axis output, however, oscillates about a line having a slope of $1/b_2$. Thus, a constant angular velocity input produces an output in the cross axis with a constantly increasing mean value. Because of this output, an ideal gyro is also called a *rate-integrating* gyro, since its long-term outputs (the pick-off angles δ_x, δ_y) are proportional to the integrals of the angular velocity components about the corresponding cross axes. (Note that the constant of proportionality for one input-output pair has the same numerical magnitude as that of the other input-output pair, but is of opposite sign.)

Since the pick-off angles (i.e., the angular displacements of the wheel plane) cannot be large in a typical gyro, a rate-integrating gyro is not suitable for applications in which the integrals of the body rates (i.e., the displacement of the gyro case relative to the rotor, whose axis tends to remain stationary in space) are appreciable. Since the motion of the craft (air, sea, or space) which carries the gyro cannot be confined to such small angles, the gyros are

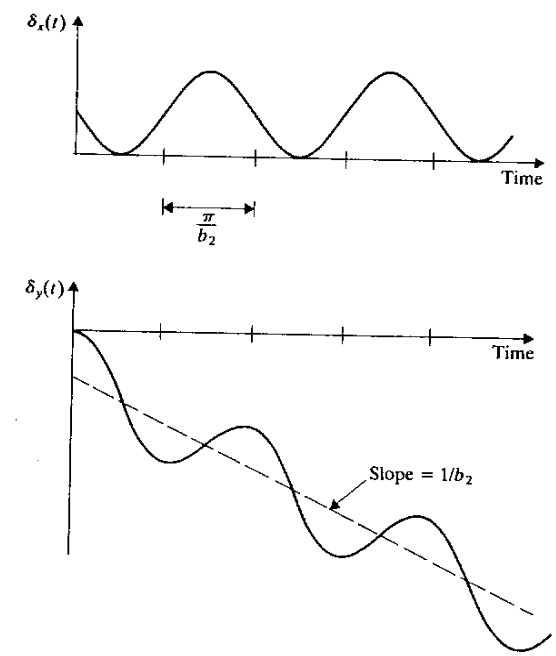

Figure 3.6 Outputs on two axes of gyro for constant angular velocity on x-axis.

typically mounted on a *stable platform* which is connected to the carrying craft by means of a set of gimbals that permit the stable platform to maintain a fixed orientation in space while the carrying craft undergoes arbitrary motion. Any tendency of the stable platform to rotate in space is immediately sensed by the gyro pick-offs and the output signals are used to generate feedback signals that drive gimbal torquers which move the gimbals to maintain the pick-off angles very close to null.

3.6 TRANSFORMATION OF STATE VARIABLES

It frequently happens that the state variables used in the original formulation of the dynamics of a system are not as convenient as another set of state variables. Instead of having to reformulate the system dynamics, it is possible to transform the matrices A, B, C, and D of the original formulation to a new set of matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$. The change of variables is represented by a linear transformation

$$z = Tx \tag{3.68}$$

where z is the state vector in the new formulation and x is the state vector in the original formulation. It is assumed that the transformation matrix T is a nonsingular k by k matrix, so that we can always write

$$x = T^{-1}z \tag{3.69}$$

We assume, moreover, that T is a constant matrix. (This assumption is not necessary, however, but the formulas to be derived below will require modification to include $\dot{T}$, if T is not constant.)

The original dynamics are expressed by

$$\dot{x} = Ax + Bu$$

and the output by

$$y = Cx + Du$$

Substitution of x as given by (3.69) into these equations gives

$$T^{-1}\dot{z} = AT^{-1}z + Bu$$

or

$$\dot{z} = TAT^{-1}z + TBu \tag{3.70}$$

$$y = CT^{-1}z + Du \tag{3.71}$$

These are in the normal form

$$\dot{z} = \bar{A}z + \bar{B}u \tag{3.72}$$

$$y = \bar{C}z + \bar{D}u \tag{3.73}$$

with

$$\bar{A} = TAT^{-1} \qquad \bar{B} = TB \qquad \bar{C} = CT^{-1} \qquad \bar{D} = D \tag{3.74}$$

In the language of matrix algebra, the dynamics matrix of the transformed system $\bar{A} = TAT^{-1}$ is said to be *similar* to the dynamics matrix A of the original system. A well-known fact of matrix algebra is that similar matrices have the same characteristic polynomial. If we didn't already know this we could show it using the argument that the input-output relations for the system, i.e., the transfer function from the input to the output, should not depend on how the state variables are defined. Using the original state variables, we found in the previous section that the transfer function is given by

$$\mathrm{H}(s) = \frac{CBs^{k-1} + CE_2Bs^{k-2} + \cdots + CE_kB}{s^k + a_1s^{k-1} + \cdots + a_k} + D \tag{3.75}$$

Using the new state variables, the transfer function is given by

$$\mathrm{H}(s) = \frac{\bar{C}\bar{B}s^{k-1} + \bar{C}\bar{E}_2\bar{B}s^{k-2} + \cdots + \bar{C}\bar{E}_k\bar{B}}{s^k + \bar{a}_1s^{k-1} + \cdots + \bar{a}_k} + \bar{D} \tag{3.76}$$

where

$$s^k + \bar{a}_1s^{k-1} + \cdots + \bar{a}_k = |sI - \bar{A}|$$

and

$$\operatorname{adj}(sI - \bar{A}) = Is^{k-1} + \bar{E}_2s^{k-2} + \cdots + \bar{E}_k$$

In order for the two transfer functions given by (3.75) and (3.76) to be equal, we need $\bar{D} = D$, which we have already determined, and we also must have

$$CB = \bar{C}\bar{B} \tag{3.77}$$

$$CE_iB = \bar{C}\bar{E}_i\bar{B} \qquad i = 1, 2, \ldots, k \tag{3.78}$$

$$a_i = \bar{a}_i \qquad i = 1, 2, \ldots, k \tag{3.79}$$

Using (3.74) $\bar{C}\bar{B} = CT^{-1}TB = CB$, so (3.77) is satisfied. The condition that $a_i = \bar{a}_i$ is a verification of the condition that the characteristic polynomials of similar matrices are equal. Finally, we must verify that (3.78) is satisfied. This is done with the aid of (3.56). For the original system, (3.56) gives

$$CE_{i+1}B = CAE_iB + a_iCB \tag{3.80}$$

and, from (3.74) $C = \bar{C}T$ and $B = T^{-1}\bar{B}$. Thus (3.80) becomes

$$\bar{C}TE_{i+1}T^{-1}\bar{B} = \bar{C}TA(T^{-1}T)E_iT^{-1}\bar{B} + \bar{a}_2\bar{C}\bar{B} \tag{3.81}$$

Note that $T^{-1}T = I$ has been inserted and that (3.79) and (3.77) have been used. It is thus seen that (3.81) reduces to

$$\bar{C}\bar{E}_{i+1}\bar{B} = \bar{C}\bar{A}\bar{E}_i\bar{B} + \bar{a}_i\bar{C}\bar{B}$$

which will satisfy (3.78) provided that

$$\bar{E}_i = TE_iT^{-1}$$

which means that each coefficient matrix E_i of the adjoint matrix of $\bar{A}$

transforms from the corresponding coefficient matrix E_i of the original matrix A, in the same way as $\bar{A}$ transforms from A, i.e.,

$$\bar{A} = TAT^{-1}$$

as given by (3.74). This is another fact of matrix algebra, which has been verified by the requirement that transfer functions between the input and the output must not depend on the definition of the state vector.

Example 3H Spring-coupled masses The equations of motion of a pair of masses M_1 and M_2 coupled by a spring, and sliding in one dimension in the absence of friction (see Fig. 3.7(*a*)) are

$$\ddot{x}_1 + \frac{K}{M_1}(x_1 - x_2) = \frac{u_1}{M_1}$$
$$\ddot{x}_2 + \frac{K}{M_2}(x_2 - x_1) = \frac{u_2}{M_2} \tag{3H.1}$$

where u_1 and u_2 are the externally applied forces and K is the spring constant. Defining the state

$$x = [x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2]'$$

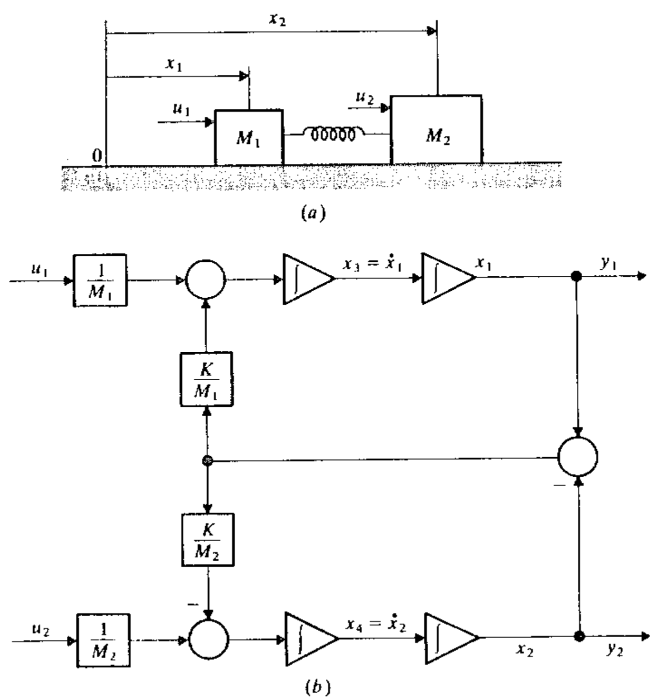

Figure 3.7 Dynamics of spring-coupled masses. (*a*) System configuration; (*b*) Block diagram.

results in the following matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/M_1 & K/M_1 & 0 & 0 \\ K/M_2 & -K/M_2 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/M_1 & 0 \\ 0 & 1/M_2 \end{bmatrix} \tag{3H.2}$$

It might be more convenient, however, to define the motion of the system by the motion of the center-of-mass

$$\bar{x} = \frac{M_1}{M}x_1 + \frac{M_2}{M}x_2 \qquad (M = M_1 + M_2) \tag{3H.3}$$

and the difference

$$\delta = x_1 - x_2 \tag{3H.4}$$

between the positions of the two masses. We let

$$z = [\bar{x}, \delta, \dot{\bar{x}}, \dot{\delta}]$$

From (3H.3) and (3H.4)

$$\dot{\bar{x}} = \frac{M_1}{M}\dot{x}_1 + \frac{M_2}{M}\dot{x}_2$$
$$\dot{\delta} = \dot{x}_1 - \dot{x}_2$$

Thus we have

$$\begin{bmatrix} \bar{x} \\ \delta \\ \dot{\bar{x}} \\ \dot{\delta} \end{bmatrix} = \underbrace{\begin{bmatrix} M_1/M & M_2/M & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & M_1/M & M_2/M \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \tag{3H.5}$$

The 4 by 4 matrix in (3H.5) is the transformation matrix T, the inverse of which is easily found to be

$$T^{-1} = \begin{bmatrix} 1 & M_2/M & 0 & 0 \\ 1 & -M_1/M & 0 & 0 \\ 0 & 0 & 1 & M_2/M \\ 0 & 0 & 1 & -M_1/M \end{bmatrix} \tag{3H.6}$$

Thus we find

$$\bar{A} = TA\bar{T}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -KM/M_1M_2 \end{bmatrix} \qquad \bar{B} = TB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/M & 1/M \\ -1/M_1 & -1/M_2 \end{bmatrix} \tag{3H.7}$$

The differential equations corresponding to $\bar{A}$ and $\bar{B}$ are

$$\ddot{\bar{x}} = \frac{u_1 + u_2}{M}$$
$$\ddot{\delta} = -\frac{KM}{M_1 M_2}\delta + \frac{u_1}{M_1} - \frac{u_2}{M_2} \tag{3H.8}$$

In this case, these equations could readily have been obtained directly from the original equations (3H.1).

3.7 STATE-SPACE REPRESENTATION OF TRANSFER FUNCTIONS: CANONICAL FORMS

In Sec. 3.5 we learned how to determine the transfer function of a linear, time-invariant system, given the state-space representation. Sometimes it is necessary to go in the other direction: from the transfer-function to the state-space representation. This need may arise because the only available description of a subsystem within a larger system is the transfer function of that subsystem. In order to use state-space methods, the transfer function must be turned into a set of first-order differential equations. Another reason for converting a transfer-function representation into a state-space representation is for the purpose of transient response simulation. Many algorithms and numerical integration computer programs designed for solution of systems of first-order equations are available, but there is not much software for numerical inversion of Laplace transforms. Thus, if a reliable method is needed for calculating the transient response of a system, one may be better off converting the transfer function of the system to state-space form and numerically integrating the resulting differential equations rather than attempting to compute the inverse Laplace transform by numerical methods.

In the last section we saw that there are innumerable systems that have the same transfer function. Hence the representation of a transfer function in state-space form is obviously not unique. In this section we shall develop several standard, or "canonical" representations of transfer functions that can always be used for single-input, multiple-output or multiple-input, single-output systems. One canonical representation has no general advantage over any other, and, moreover, there is no reason why a canonical representation is to be preferred over a noncanonical representation.

First companion form The development starts with a transfer function of a single-input, single-output system of the form

$$H(s) = \frac{y(s)}{u(s)} = \frac{1}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{3.82}$$

which can be written

$$(s^k + a_1 s^{k-1} + \cdots + a_k)y(s) = u(s) \tag{3.83}$$

The differential equation corresponding to (3.83) is

$$D^k y + a_1 D^{k-1} y + \cdots + a_k y = u \tag{3.84}$$

where $D^k y$ stands for $d^k y/dt^k$. Solve for the highest derivative in (3.84)

$$D^k y = -a_1 D^{k-1} y - a_2 D^{k-2} y - \cdots - a_k y + u \tag{3.85}$$

Now consider a chain of k integrators as shown in Fig. 3.8(a), and suppose that the output of the last integrator is y. Then the output of the next-to-last integrator is $Dy = dy/dt$, and so forth. The output from the first integrator is

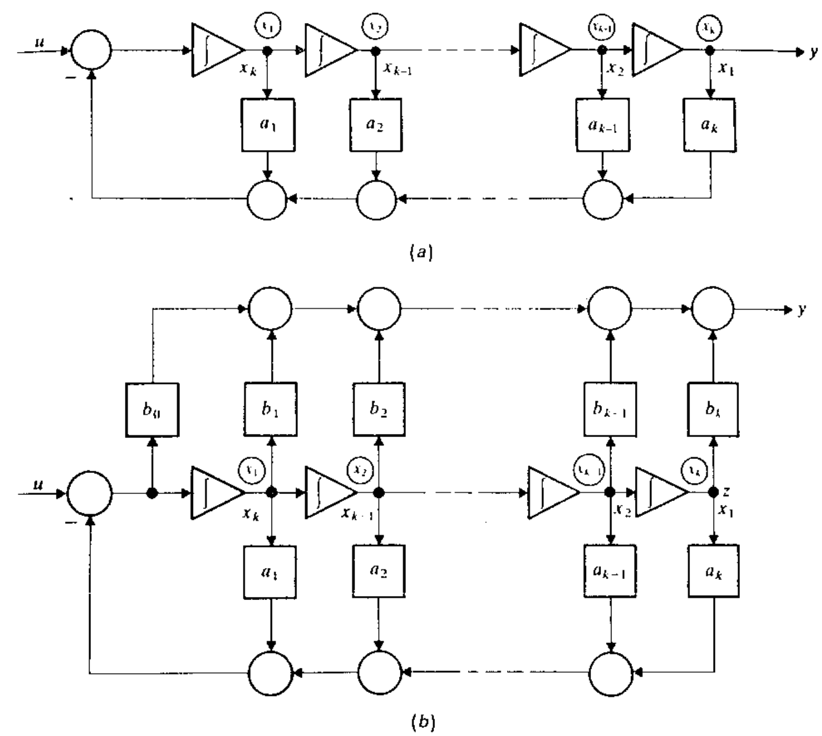

Figure 3.8 State-space realization of transfer functions in first companion form

(a) $H(s) = \dfrac{1}{s^k + a_1 s^{k-1} + \cdots + a_k}$ (b) $H(s) = \dfrac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k}$

$D^{k-1}y$ and the input to this integrator is thus $D^k y$. From (3.85) it follows that Fig. 3.8(a) represents the given transfer function (3.82) provided that the feedback gains are chosen as shown in the figure. To get one state-space representation of the system, we identify the output of each integrator with a state variable, starting at the right and proceeding to the left. The corresponding differential equations using this identification of state variables are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots\cdots\cdots\cdots \\ \dot{x}_{k-1} &= x_k \\ \dot{x}_k &= -a_k x_1 - a_{k-1} x_2 - \cdots - a_1 x_k + u \end{aligned} \tag{3.86}$$

The output equation is simply

$$y = x_1 \tag{3.87}$$

The matrices corresponding to (3.86) and (3.87) are

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.88}$$

$$C = [1 \quad 0 \quad 0 \quad \cdots \quad 0]$$

The matrix A has a very special structure: the coefficients of the denominator of the transfer function, preceded by minus signs, are strung out along the bottom row of the matrix. The rest of the matrix is zero except for the "superdiagonal" terms which are all unity. In matrix theory, a matrix with this structure is said to be *in companion form*. For this reason we identify this state-space realization of the transfer function as a *companion-form* realization. We call this the *first companion form*; another companion form will be discussed later on.

If the state variables were numbered from right to left we would have

$$\begin{aligned} \dot{x}_1 &= -a_1x_1 - a_2x_2 - \cdots - a_kx_k + u \\ \dot{x}_2 &= x_1 \\ &\cdots\cdots\cdots \\ \dot{x}_{k-1} &= x_{k-2} \\ \dot{x}_k &= x_{k-1} \end{aligned} \tag{3.89}$$

and

$$y = x_k$$

The corresponding matrices would be

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.90}$$

$$C = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

This representation is also called a companion form, but is less frequently used than the form (3.88). There is nothing sacred about numbering the integrators systematically from right to left or from left to right. A perfectly valid, if perverse, representation would result if the integrators were numbered at random.

Having developed a state-space representation of the simple transfer function (3.82), we are now in a position to consider the more general transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{b_0s^k + b_1s^{k-1} + \cdots + b_k}{s^k + a_1s^{k-1} + \cdots + a_k} \tag{3.91}$$

The development is aided by the introduction of an intermediate variable $z(s)$

$$\frac{y(s)}{u(s)} = \frac{y(s)}{z(s)}\frac{z(s)}{u(s)} = \frac{b_0s^k + b_1s^{k-1} + \cdots + b_k}{s^k + a_1s^{k-1} + \cdots + a_k}$$

We identify the first factor with the numerator and the second factor with the denominator:

$$\frac{y(s)}{z(s)} = b_0s^k + b_1s^{k-1} + \cdots + b_k \tag{3.92}$$

and

$$\frac{z(s)}{u(s)} = \frac{1}{s^k + a_1s^{k-1} + \cdots + a_k} \tag{3.93}$$

The realization of the transfer function from u to z has already been developed. And, from (3.92)

$$y(s) = (b_0s^k + b_1s^{k-1} + \cdots + b_k)z(s)$$

i.e.,

$$y = b_0D^kz + b_1D^{k-1}z + \cdots + b_kz$$

The inputs to the integrators in the chain are the k successive derivatives of z as shown in Fig. 3.8(b), hence we have the required state-space representation. All that remains to be done is to write the corresponding differential equations. The state equations are the same as (3.86) or (3.89) and hence the A and B matrices are the same. The output equation is found by careful examination of the block diagram of Fig. 3.8(b). Note that there are *two* paths from the output of each integrator to the system output: one path upward through the box labeled b_i and a second path down through the box labeled a_i and thence through the box labeled b_0. As a consequence, when the right-to-left state variable numbering is used

$$y = (b_k - a_kb_0)x_1 + (b_{k-1} - a_{k-1}b_0)x_2 + \cdots + (b_1 - a_1b_0)x_k + b_0u$$

Hence

$$C = [b_k - a_kb_0,\ b_{k-1} - a_{k-1}b_0, \ldots, b_1 - a_1b_0], \qquad D = [b_0] \tag{3.94}$$

If the direct path through b_0 is absent, then the D matrix is zero and the C matrix contains only the b_i coefficients.

If left-to-right numbering is used, then

$$C = [b_1 - a_1 b_0, b_2 - a_2 b_0, \ldots, b_k - a_k b_0], \qquad D = [b_0] \tag{3.95}$$

The structure of the first canonical form is very easy to remember ("automnemonic"). The string of integrators can be visualized as the fraction bar of the transfer function (3.91) that is realized. The numerator coefficients appear above the chain of integrators in the same order as they appear above the fraction bar in (3.91) and the denominator coefficients appear below the chain of integrators in the same order as they appear below the fraction bar in (3.91). Not too much imagination is needed to "see" the transfer function (3.91) in Fig. 3.8.

A generalized version of the first companion form can be used to realize a single input, multiple output system represented by l transfer functions, one from the single input to each of the l different outputs

$$\frac{y_1(s)}{u(s)} = \frac{b_{01}s^k + b_{11}s^{k-1} + \cdots + b_{k1}}{s^k + a_1 s^{k-1} + \cdots + a_k}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{y_l(s)}{u(s)} = \frac{b_{0l}s^k + b_{1l}s^{k-1} + \cdots + b_{kl}}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

The same set of state variables serves for each transfer function. Each numerator, however, is realized by a different set of gains, as shown in Fig. 3.9. Thus the A and B matrices are exactly as given earlier. From Fig. 3.9 it is also seen that the C and D matrices are

$$C = \begin{bmatrix} b_{k1} - a_k b_{01} & b_{k-1,1} - a_{k-1} b_{01} & \cdots & b_{11} - a_1 b_{01} \\ \cdots & \cdots & \cdots & \cdots \\ b_{kl} - a_k b_{0l} & b_{k-1,l} - a_{k-1} b_{0l} & \cdots & b_{1l} - a_1 b_{0l} \end{bmatrix} \qquad D = \begin{bmatrix} b_{01} \\ \vdots \\ b_{0l} \end{bmatrix} \tag{3.96}$$

for the right-to-left numbering, or

$$C = \begin{bmatrix} b_{11} - a_1 b_{01} & b_{21} - a_2 b_{01} & \cdots & b_{k1} - a_k b_{01} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1l} - a_1 b_{0l} & b_{2l} - a_2 b_{0l} & \cdots & b_{kl} - a_k b_{0l} \end{bmatrix} \qquad D = \begin{bmatrix} b_{0l} \\ \vdots \\ b_{0l} \end{bmatrix} \tag{3.97}$$

for the left-to-right numbering.

In the first canonical form realizations of Figs. 3.1 through 3.9 the input is connected directly to the first integrator in the chain and the output is a linear combination of the outputs of the integrators (and the input, when the D matrix is nonzero). This form is useful not only for single-input, single-output systems, but also, as we have seen, for single-input, multiple-output systems. A variant of the structure of Fig. 3.8, in which the output is taken directly from the last integrator but the input is connected to all the integrators, is shown in Fig. 3.10. A realization of a multiple-input, single-output system based on the structure of Fig. 3.10 is shown in Fig. 3.11.

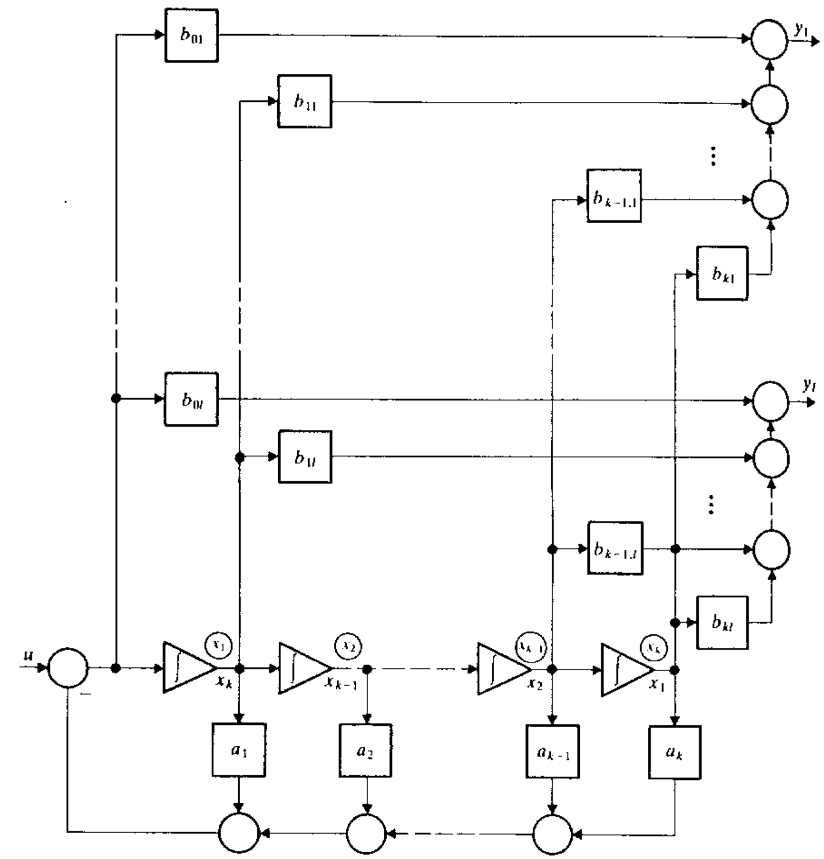

Figure 3.9 Realization of single-input, multiple-output system in first companion form.

The "feedforward" gains $p_1, p_2, \ldots, p_k$ in Fig. 3.10 are in general not equal to the coefficients $b_1, b_2, \ldots, b_k$ of the transfer function but must be obtained by solution of a set of linear algebraic equations which may be derived as follows. From Fig. 3.10 it is easy to see that

$$\begin{aligned} \dot{x}_1 &= x_2 + p_1 u \\ \dot{x}_2 &= x_3 + p_2 u \\ &\cdots\cdots\cdots\cdots \\ \dot{x}_{k-1} &= x_k + p_{k-1} u \\ \dot{x}_k &= -a_1 x_k - \cdots - a_k x_1 + p_k u \end{aligned} \tag{3.98}$$

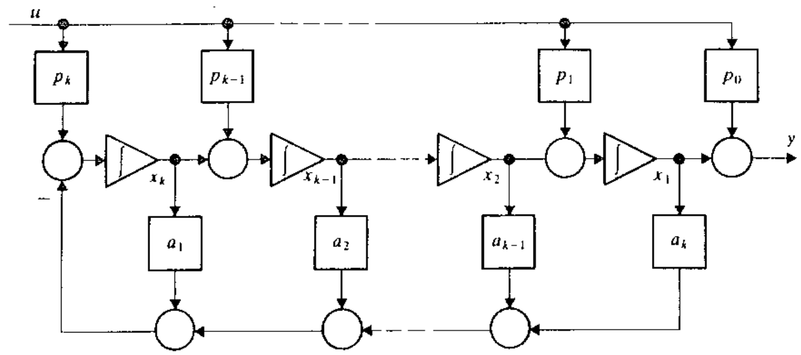

Figure 3.10 Alternative first companion form of realization of transfer function

$$H(s) = \frac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

and

$$y = x_1 + p_0 u \tag{3.99}$$

Differentiate (3.99) k times and use (3.98) to obtain

$$\begin{aligned} Dy &= x_2 + p_1 u + p_0 Du \\ D^2 y &= x_3 + p_2 u + p_1 Du + p_2 D^2 u \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ D^{k-1} y &= x_k + p_{k-1} u + p_{k-2} Du + \cdots + p_1 D^{k-2} u + p_0 D^{k-1} u \\ D^k y &= -a_1 x_k - a_2 x_{k-1} - \cdots - a_k x_1 + p_k u + p_{k-1} Du + \cdots \\ &\quad + p_1 D^{k-1} u + p_0 D^k u \end{aligned} \tag{3.100}$$

From (3.100) and (3.99) we thus get

$$\begin{aligned} D^k y + a_1 D^{k-1} y + \cdots + a_{k-1} Dy + a_k y \\ = (p_k + a_1 p_{k-1} + \cdots + a_{k-1} p_1 + \quad a_k p_0) u \\ + \quad (p_{k-1} + \cdots + a_{k-2} p_1 + a_{k-1} p_0) Du \\ + \cdots + (p_1 + a_1 p_0) D^{k-1} u \\ p_0 D^k u \end{aligned} \tag{3.101}$$

In order for (3.101) to represent the differential equation corresponding to the

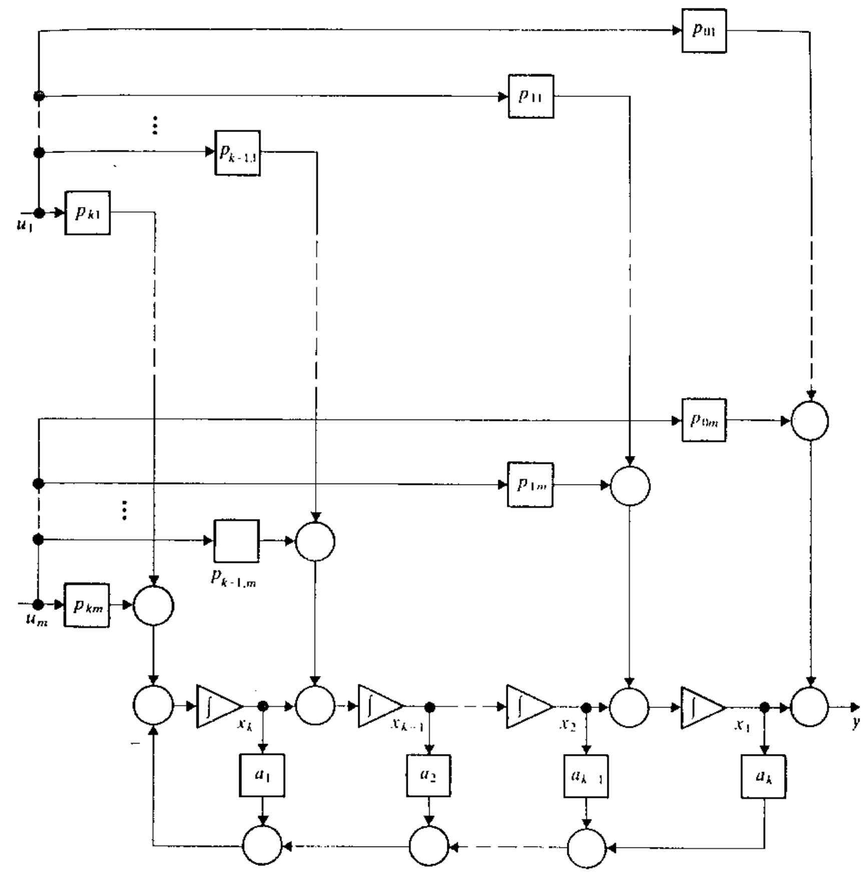

Figure 3.11 Use of alternative first companion form for realizing multiple-input single-output transfer function.

transfer function (3.91) it is necessary that

$$\begin{aligned} p_0 &= b_0 \\ p_1 + a_1 p_0 &= b_1 \\ &\cdots\cdots\cdots\cdots \\ p_{k-1} + \cdots + a_{k-2} p_1 + a_{k-1} p_0 &= b_{k-1} \\ p_k + \cdots + a_{k-1} p_1 + a_k p_0 &= b_k \end{aligned} \tag{3.102}$$

which constitute a set of $k+1$ simultaneous equations for $p_0, p_1, \ldots, p_k$. These may be arranged in vector-matrix form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-1} & a_{k-2} & \cdots & 0 \\ a_k & a_{k-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \\ p_k \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \\ b_k \end{bmatrix} \quad (3.103)$$

The triangular matrix that appears in (3.103), the first column of which is formed from the coefficients of the characteristic polynomial, and whose subsequent columns are obtained by pushing down the previous column one position, is a special form of a *Toeplitz matrix*, and occurs elsewhere in linear system theory. We shall encounter it again in Chap. 6 in connection with control system design by pole placement. The determinant of this matrix is 1, so it is nonsingular. Hence it is always possible to solve for the p_i given the numerator coefficients b_i $(i = 1, 2, \ldots, k)$.

It is worth noting that although the state variables in the original first canonical form and in the alternate canonical form are identified with the outputs of the integrators, they are not the same variables: (3.86) and (3.87) are not the same as (3.98) and (3.99). Although the A matrix of both systems are the same, the B and C matrices are not. The reader might wish to test the comprehension of state-variable transformations, as discussed in the previous section, by finding the transformation matrix T that transforms (3.86) and (3.87) into (3.98) and (3.99). Note that this matrix must satisfy

$$TAT^{-1} = A \qquad \text{or} \qquad TA = AT$$

Thus T commutes with A.

The generalization of Fig. 3.10 for multiple-input, single-output systems is shown in Fig. 3.11. The set of coefficients $p_{0i}, p_{1i}, \ldots, p_{ki}$ for the ith input is found from the corresponding coefficients $b_{0i}, b_{1i}, \ldots, b_{ki}$ by use of (3.103).

By use of the structure shown in Fig. 3.9 we can realize a single-input, multiple-output system in state-variable form. Similarly, a single-output, multiple-input system can be realized with the structure of Fig. 3.11. One might think that a multiple-input, multiple-output system can be realized with only k integrators using a combination of Figs. 3.9 and 3.11. A bit of reflection, however, will soon convince one that in general this is not possible. It is obvious, however, that one way of realizing a multiple-input, multiple-output system is by using a number of structures of the form of Fig. 3.9 or Fig. 3.11 in parallel. If the number l of outputs is smaller than the number m of inputs, then l structures of Fig. 3.11 are used in parallel; if the number of outputs is greater than the number of inputs then m structures of the form of Fig. 3.9 are used. Hence it is always possible to realize an m-input, l-output system with no more than $k \cdot \min(l, m)$ integrators. But there is no assurance that there is not a realization that requires still fewer integrators. The determination of a "minimum" realization was the subject of considerable research during the 1970s. There are now several algorithms for finding a minimum realization and the matrices A, B, C, and D that result. (See Note 3.5 for a more complete discussion of this subject.)

Second companion form In the first companion form, the coefficients of the denominator of the transfer function appear in one of the rows of the A matrix. There is another set of companion forms in which the coefficients appear in a column of the A matrix. For a single-input, single-output system, this form can be obtained by writing (3.91) as

$$(s^k + a_1 s^{k-1} + \cdots + a_k)y(s) = (b_0 s^k + b_1 s^{k-1} + \cdots + b_k)u(s)$$

or

$$s^k[y(s) - b_0 u(s)] + s^{k-1}[a_1 y(s) - b_1 u(s)] + \cdots + [a_k y(s) - b_k u(s)] = 0$$

On dividing by s^k and solving for $y(s)$, we obtain

$$y(s) = b_0 u(s) + \frac{1}{s}[b_1 u_1(s) - a_1 y(s)] + \cdots + \frac{1}{s^k}[b_k u(s) - a_k y(s)] \quad (3.104)$$

Noting that the multiplier $1/s^j$ is the transfer function of a chain of j integrators, immediately leads to the structure shown in Fig. 3.12. The signal y is fed back to each of the integrators in the chain and the signal u is fed forward. Thus the signal $b_k u - a_k y$ passes through k integrators, as required by (3.104), the signal $b_{k-1}u - a_{k-1}y$ passes through $k-1$ integrators, and so forth to complete the realization of (3.104). The structure retains the ladder-like shape of the first companion form, but the feedback paths are in different directions.

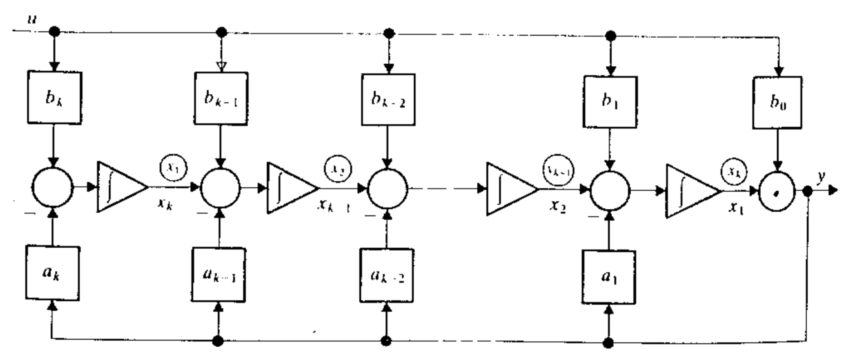

Figure 3.12 State-space realization of transfer function

$$H(s) = \frac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

in second companion form.

Using the right-to-left numbering of state variables, the differential equations corresponding to Fig. 3.12 are

$$\begin{aligned}\dot{x}_1 &= x_2 - a_1(x_1 + b_0 u) + b_1 u \\ \dot{x}_2 &= x_3 - a_2(x_1 + b_0 u) + b_2 u \\ \dot{x}_{k-1} &= x_k - a_{k-1}(x_1 + b_0 u) + b_{k-1} u \\ \dot{x}_k &= -a_k(x_1 + b_0 u) + b_k u\end{aligned} \tag{3.105}$$

and the output equation is

$$y = x_1 + b_0 u$$

Thus the matrices that describe the state-space realization are given by

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{k-1} & 0 & 0 & \cdots & 0 \\ -a_k & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_{k-1} - a_{k-1} b_0 \\ b_k - a_k b_0 \end{bmatrix} \tag{3.106}$$

$$C = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \qquad D = [b_0]$$

If the right-to-left numbering convention is employed, then instead of (3.106) we obtain

$$A = \begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} b_k - a_k b_0 \\ b_{k-1} - a_{k-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} \tag{3.107}$$

$$C = [0 \quad 0 \quad \cdots \quad 1] \qquad D = [b_0]$$

Compare the matrices A, B, C, and D with the matrices of the first companion form and observe that the A matrix of one companion form corresponds to the transpose of an A matrix of the other, and that the B and C matrices of one correspond to the transposes of the C and B matrices, respectively of the other.

The state space realization of Fig. 3.12 for a single-input, single-output system can readily be generalized to a multiple-input, multiple-output system; the upper part of the block diagram representing the realization would have the same general form as the upper part of Fig. 3.11, with one path from every input to the summer in front of each integrator. The gains are obtained from the elements of the B matrix.

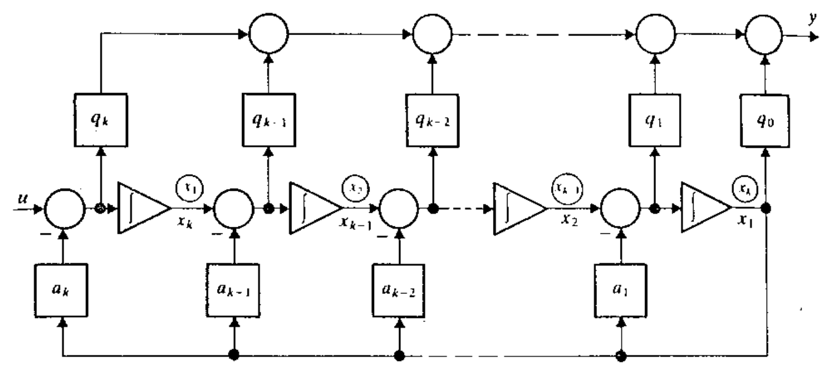

Figure 3.13 Alternate second companion form realization of transfer function.

Just as there are two versions of the first companion form, there are two versions of the second companion form. The second version has the structure shown in Fig. 3.13. The reader by now can probably guess the relationships between the gains $q_1, \ldots, q_k$ and the coefficients of the numerator of the transfer function. (See Problem 3.5.) It is also noted that the structure of Fig. 3.13 can be generalized to the realization of a single-input, multiple-output system.

Jordan Form: Partial Fraction Expansion

Another of the canonical forms of the realization of a transfer function is the *Jordan form*, so named because of the nature of the A matrix that results. This canonical form follows directly from the partial fraction expansion of the transfer functions.

The results are simplest when the poles of the transfer function are all different—no repeated poles. The partial fraction expansion of the transfer function then has the form

$$H(s) = b_0 + \frac{r_1}{s - s_1} + \frac{r_2}{s - s_2} + \cdots + \frac{r_k}{s - s_k} \tag{3.108}$$

The coefficients r_i $(i = 1, 2, \ldots, k)$ are the residues of the reduced transfer function $H(s) - b_0$ at the corresponding poles. In the form of (3.108) the transfer function consists of a direct path with gain b_0, and k first-order transfer functions in parallel. A block diagram representation of (3.108) is shown in Fig. 3.14. The gains corresponding to the residues have been placed at the outputs of the integrators. This is quite arbitrary. They could have been located on the input sides, or indeed split between the input and the output.

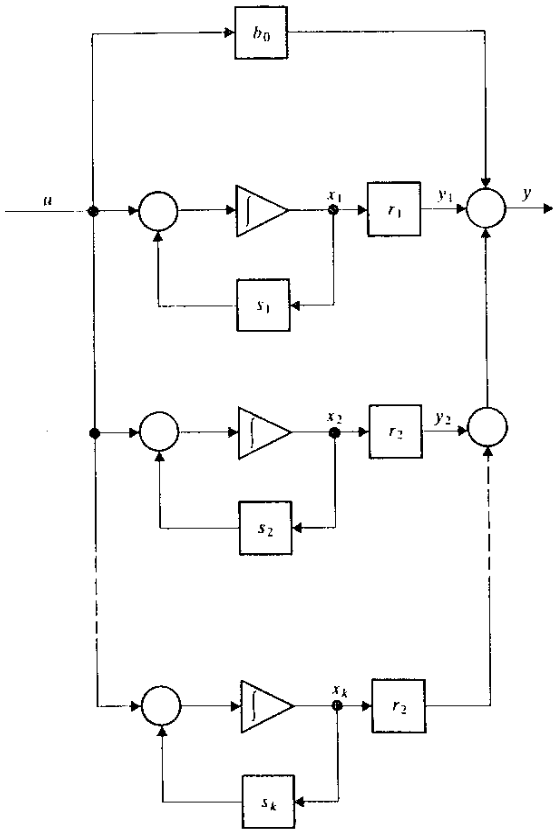

Figure 3.14 Complex Jordan form of transfer function with distinct roots.

Identifying the outputs of the integrators with the state variables results in the following differential equations:

$$\begin{aligned} \dot{x}_1 &= s_1 x_1 + u \\ \dot{x}_2 &= s_2 x_2 + u \\ &\cdots\cdots\cdots \\ \dot{x}_k &= s_k x_k + u \end{aligned} \tag{3.109}$$

and an observation equation

$$y = r_1 x_1 + r_2 x_2 + \cdots + r_k x_k + b_0 u \tag{3.110}$$

Hence the matrices corresponding to this realization are

$$A = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & s_k \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [r_1 \quad r_2 \quad \cdots \quad r_k] \qquad D = [b_0]$$

Note that A is a diagonal matrix, which in matrix theory is the Jordan form of a matrix having nonrepeated eigenvalues.

The block-diagram representation of Fig. 3.14 can be turned into hardware only if all the poles $s_1, s_2, \ldots, s_k$ are real. If they are complex, the feedback gains and the gains corresponding to the residues are complex. In this case the representation must be considered as being purely conceptual: valid for theoretical studies, but not physically realizable. If a physically realizable representation is desired, it is possible to combine a pair of complex poles and residues into a single second-order transfer function with real coefficients. The resulting second-order transfer function of the subsystem is then realized in one of the companion forms. Suppose, for example, that s_1 and s_2 are a complex conjugate pair. For a transfer function having real coefficients (as it must in a real system), the residues at a pair of complex conjugate poles must be themselves complex conjugates. Thus a pair of complex conjugate poles, say $s_1 = -\sigma + j\omega$ and $s_2 = -\sigma - j\omega$ with corresponding residues $r = \lambda + j\gamma$ and $r_2 = \lambda - j\gamma$ give rise to the sum

$$\mathrm{H}_{1,2} = \frac{\lambda + j\gamma}{s + \sigma - j\omega} + \frac{\lambda - j\gamma}{s + \sigma + j\omega} = \frac{2[\lambda s + (\lambda\sigma - \omega\gamma)]}{s^2 + 2\sigma s + \sigma^2 + \omega^2}$$

This is a second-order transfer function having the companion-form realization shown in Fig. 3.15. This will give rise to a second-order system in state-space form

$$\begin{aligned} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -(\sigma^2 + \omega^2) & -2\sigma \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y_{1,2} &= [2(\lambda\sigma - \omega\gamma) \quad 2\lambda] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{aligned} \tag{3.111}$$

A second-order subsystem such as (3.111) can be used to represent every complex conjugate pair of terms in the partial fraction expansion.

When the system has repeated roots, the partial fraction expansion of the transfer function H(s) will not be as simple as (3.109). Instead it will be of the form

$$\mathrm{H}(s) = b_0 + \mathrm{H}_1(s) + \cdots + \mathrm{H}_k(s) \tag{3.112}$$

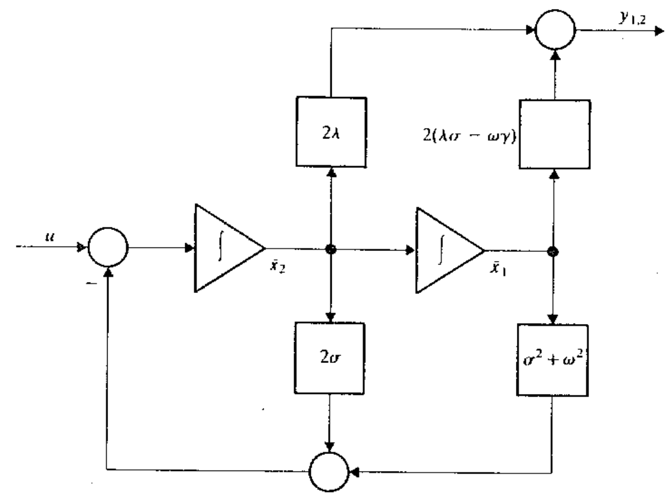

Figure 3.15 Companion-form realization of pair of complex conjugate terms as a real second-order subsystem.

where $\bar{k} < k$ is the number of *distinct* poles of H(s) and where

$$H_i(s) = \frac{r_{1i}}{s - s_i} + \frac{r_{2i}}{(s - s_i)^2} + \cdots + \frac{r_{\nu_i i}}{(s - s_i)^{\nu_i}}$$

where ν_i is the multiplicity of the ith pole ($i = 1, 2, \ldots, \bar{k}$). The last term in $H_i(s)$ can be synthesized as a chain of ν_i identical, first-order systems, each having transfer function $1/(s - s_i)$. The preceding terms in the chain of fewer than ν_i of such transfer functions. Thus the entire transfer function $H_i(s)$ can be synthesized by the system having the block diagram in Fig. 3.16.

Using the right-to-left numbering convention gives the differential equations

$$\begin{aligned} \dot{x}_{1i} &= s_i x_{1i} + u \\ \dot{x}_{2i} &= x_{1i} + s_i x_{2i} \\ &\cdots\cdots\cdots\cdots \\ \dot{x}_{\nu_i i} &= x_{(\nu_i - 1)i} + s_i x_{\nu_i i} \end{aligned} \tag{3.113}$$

and the output is given by

$$y_i = r_{1i}x_{1i} + r_{2i}x_{2i} + \cdots + r_{\nu_i i}x_{\nu_i i} \tag{3.114}$$

If the state vector for the subsystem is defined by

$$x^i = [x_{1i} \quad x_{2i} \quad \cdots \quad x_{\nu_i i}]'$$

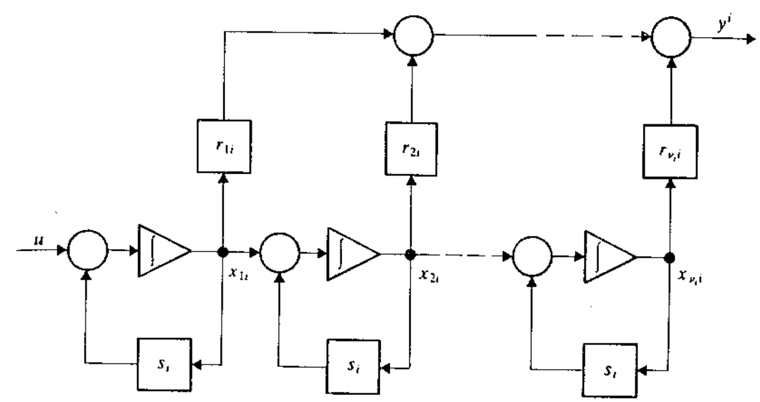

Figure 3.16 Jordan-block realization of part of transfer function having repeated pole

$$H_i(s) = \frac{r_{1i}}{s - s_i} + \cdots + \frac{r_{\nu_i i}}{(s - s_i)^{\nu_i}}$$

then (3.113) and (3.114) can be written in the standard form

$$\begin{aligned} \dot{x}^i &= A_i x^i + b_i u \\ y^i &= C_i x^i \end{aligned} \tag{3.115}$$

where

$$A_i = \begin{bmatrix} s_i & 0 & 0 & \cdots & 0 \\ 1 & s_i & 0 & \cdots & 0 \\ 0 & 1 & s_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & s_i \end{bmatrix} \qquad B_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.116}$$

$$C_i = [r_{1i} \quad r_{2i} \quad 0 \quad \cdots \quad r_{\nu_i i}]$$

Note that the A matrix of the subsystem consists of two diagonals: the principal diagonal has the corresponding characteristic root (pole) and the subdiagonal has all 1's. In matrix theory a matrix having this structure is said to be *in Jordan form*, which is the name used for this realization of the transfer function.

If the right-to-left numbering convention were employed it is easy to see that the A matrix would have 1's on the superdiagonal instead of on the subdiagonal. This is an alternate Jordan form.

According to (3.112) the overall transfer function consists of a direct path with gain b_0 and $\bar{k}$ subsystems, each of which is in the Jordan canonical form, as shown in Fig. 3.17. The state vector of the overall system consists of the

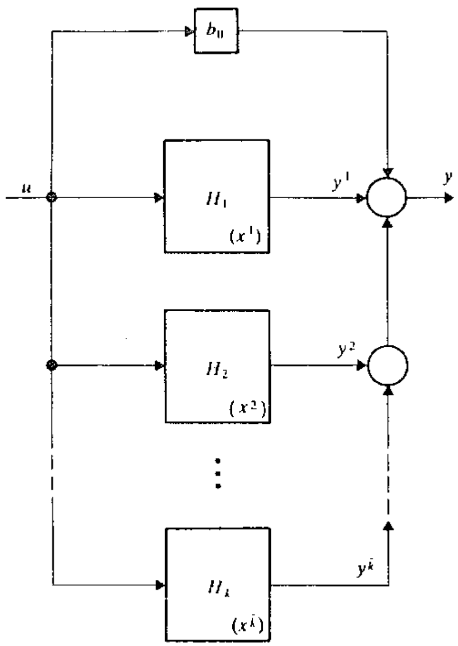

Figure 3.17 Subsystems in Jordan canonical form combined into overall system.

concatenation of the state vectors of each of the Jordan blocks

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix} \tag{3.117}$$

Since there is no feedback from any of the subsystems to the others, the A matrix of the overall system is "block diagonal":

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \tag{3.118}$$

where each of the submatrices is in the Jordan canonical form shown in (3.116). The B and C matrices of the overall system are the concatenations of the B_i and C_i matrices of each of the subsystems:

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix} \qquad C = [C_1, \ldots, C_k] \tag{3.119}$$

It is noted that the Jordan blocks are only conceptual if the poles are complex. Pairs of Jordan blocks can be combined to give a real Jordan block of order $2\nu_i$. The details are easy to work out, but the general notation and calculations are quite messy. If the need ever arises (which is highly unlikely) for such a real Jordan block, the procedures used in this section can be followed to obtain the required result.

To conclude this discussion it is noted that the Jordan normal form can be extended directly to either a multiple-input, single-output system, or a multiple-output, single-input system. In the former case, each input has a path to each of the integrators; in the latter, each integrator has a path to each of the outputs.

Example 31. Spring-coupled masses (continued) It is readily established, either by use of the general relationship (3.65) applied to (3H.2), or by simpler means, that the input-output relationship for the spring-coupled mass system is given by

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s^2 + K/M_2}{s^2(s^2 + K/\bar{M})} & \dfrac{K/M_1}{s^2(s^2 + K/\bar{M})} \\ \dfrac{K/M_2}{s^2(s^2 + K/\bar{M})} & \dfrac{s^2 + K/M_1}{s^2(s^2 + K/\bar{M})} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \tag{31.1}$$

where

$$\frac{1}{\bar{M}} = \frac{1}{M_1} + \frac{1}{M_2}$$

The block diagram of Fig. 3.7 already gives a state-variable realization of the system. For illustrative purposes, however, we assume that $u_2 = 0$, and hence we have a single-input, two-output system. The transfer functions of interest are

$$H_1(s) = \frac{y_1(s)}{u_1(s)} = \frac{s^2 + K/M_2}{s^2(s^2 + K/\bar{M})}$$
$$H_2(s) = \frac{y_2(s)}{u_1(s)} = \frac{K/M_2}{s^2(s^2 + K/\bar{M})} \tag{31.2}$$

The first companion form, using the structure of Fig. 3.9 for a single-input, multiple-output, system is obtained directly from (31.2) and is shown in Fig. 3.18(a). The corresponding matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K/\bar{M} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} K/M_2 & 0 & 1 & 0 \\ K/M_2 & 0 & 0 & 0 \end{bmatrix} \qquad D = 0 \tag{31.3}$$

Although the structure and gains for the single-input, multiple-output version of the second companion form were not given explicitly, it is readily established that the block diagram of Fig. 3.18(b) correctly represents the transfer functions from u_1 to y_1 and y_2. The relevant matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/\bar{M} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{31.4}$$
$$C = \begin{bmatrix} -K/M_1 & 0 & 1 & 0 \\ 0 & K/M_2 & 0 & 0 \end{bmatrix} \qquad D = 0$$

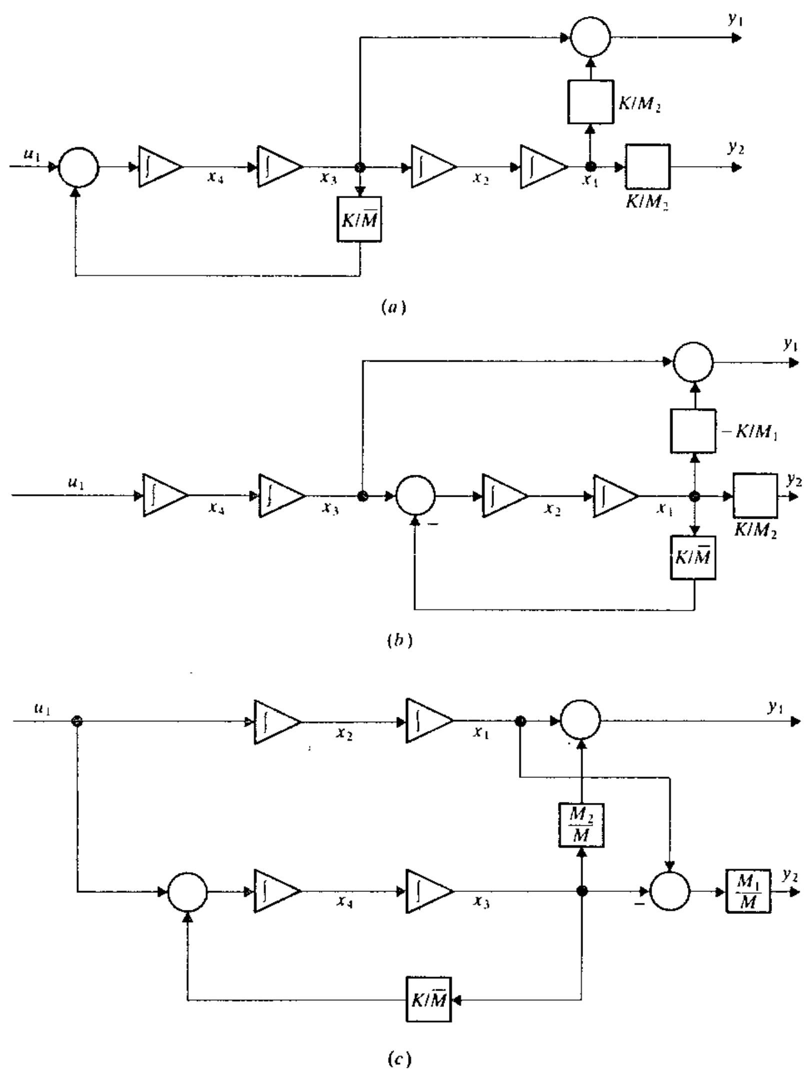

Figure 3.18 Canonical realizations of transfer functions of spring-coupled mass system. (*a*) First companion form, (*b*) second companion form, (*c*) Jordan canonical form.

To obtain the Jordan canonical form we expand the transfer functions in partial fractions

$$H_1(s) = \frac{1}{s^2} + \frac{M_2 M}{s^2 + K/\bar{M}}$$
$$H_2(s) = \frac{M_1/M}{s^2} + \frac{M_1/M}{s^2 + K/\bar{M}} \qquad (M = M_1 + M_2) \qquad (31.5)$$

The system has a double pole at the origin and a pair of imaginary poles at $s = \pm j\sqrt{K/\bar{M}}$. To the real form, the two terms with the imaginary poles are already combined in (31.5). The block diagram representation of (31.5) in the form appropriate for a single-input, two-output system is shown in Fig. 3.18(*c*). The system matrices corresponding to this realization are

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -K/\bar{M} & 0 \end{array}\right] \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & M_2/M & 0 \\ M_1/M & 0 & -M_1/M & 0 \end{bmatrix} \qquad D = 0$$

The *A* matrix has been partitioned to show the block-diagonal form. The upper left-hand matrix is in the (superdiagonal) Jordan form for a repeated pole at the origin; the lower right-hand matrix is in the companion form for a second-order system.

PROBLEMS

Problem 3.1 Exercises in resolvents and transition matrices

Find the resolvents and transition matrices for each of the following:

(*a*) $A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 2 & -3 \end{bmatrix}$

(*b*) $A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

(*c*) $A_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

Problem 3.2 Exercises on canonical forms

Determine the canonical forms (companion and Jordan) for each of the following transfer functions:

(*a*) $H(s) = \dfrac{(s+2)(s+4)}{(s+1)(s+3)(s+5)}$

(*b*) $H(s) = \dfrac{s+2}{s[(s+1)^2+4]}$

(*c*) $H(s) = \dfrac{s+3}{(s+1)^2(s+2)}$

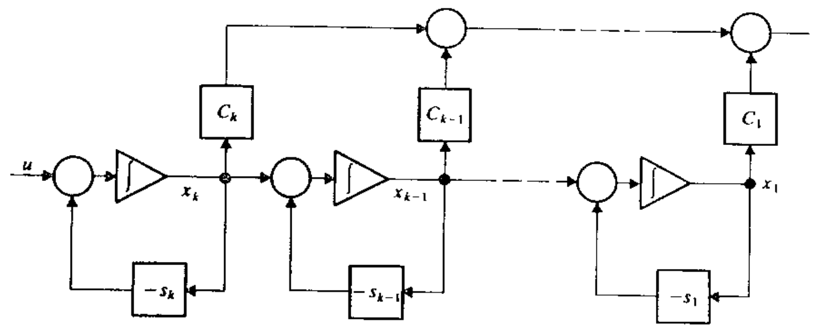

Figure P3.3 Tandem canonical form.

Problem 3.3 Another canonical form

An alternative to the Jordan canonical form for single-input, single-output systems is the "tandem form" shown in Fig. P3.3.

(*a*) Write the A, B, and C matrices for this form.

(*b*) Given the system in Jordan form $\dot{x} = \Lambda x + Bu$ where $\Lambda = \text{diag}[-s_1, -s_2, \ldots, -s_k]$, find the transformation matrix T that transforms it to the tandem form.

Problem 3.4 Adjoint equation

Show that the state transition matrix satisfies the following differential equation

$$\frac{\partial \Phi(t, \tau)}{\partial \tau} = -\Phi(t, \tau)A(\tau) \tag{P3.4}$$

Hint: Use $dX^{-1}(t)/dt = -X^{-1}(t)(dX(t)/dt)X^{-1}(t)$.

Equation (P3.4) is sometimes called the "adjoint" equation, or the "backward-evolution" equation.

Problem 3.5 Coefficients in second companion form

Find the relationship between the coefficients $q_1, \ldots, q_k$ of the second companion form, Fig. 3.13, to the coefficients of the numerator and denominator of the transfer function $H(s)$.

Problem 3.6 Motor-driven cart with pendulum

Consider the inverted pendulum on a cart driven by an electric motor that was studied in Prob. 2.1. Let the state vector, control, and outputs be defined by

$$x = [x, \dot{x}, \theta, \dot{\theta}]' \qquad u = e \qquad y = [x, \theta]'$$

(*a*) Find the matrices A, B, C, and D of the state-space characterization of the system.

(*b*) Draw the block-diagram representation of the system.

(*c*) Find the resolvent and the state-transition matrix.

(*d*) Find the transfer functions from the input u to the two outputs.

The following numerical data may be used if you would rather use numbers than letters:

$$m = 0.1\text{ kg} \qquad M = 1.0\text{ kg} \qquad l = 1.0\text{ m} \qquad g = 9.8\text{ m}\cdot\text{s}^{-2}$$

$$k = 1\text{ V}\cdot\text{s} \qquad R = 100\,\Omega \qquad r = 0.02\text{ m}$$

Problem 3.7 Three-capacitance thermal system

For the insulated conducting bar of Prob. 2.1, using as the state, vector, control, and exogenous variables

$$x = [v_1, v_2, v_3]'$$

$$u = e_0$$

$$x_0 = v_0$$

(*a*) Find the matrices A, B, and E of the state-space characterization of the system.

(*b*) Find the resolvent and the state-transition matrix.

(*c*) Find the transfer function from the input $u = e_0$ to the output $y = v_3$.

Use $R = 1$, $C = 2$.

Problem 3.8. Eigenvalues of R–C network

Consider a *passive* electrical network (consisting of only capacitors, resistors, plus voltage, and current sources). Show that all the eigenvalues lie on the *negative real* axis.

Problem 3.9 Two-car train

Consider the two-car train of Prob. 2.5 with the following numerical data:

Trains: $M_1 = M_2 = 1.0$ kg, $K = 40$ N/m.
Motors: $k = 2$ V · s, $R = 100\,\Omega$, $r = 2$ cm.

(*a*) Find the transfer functions from the input voltages to the motor positions.

(*b*) Find the open-loop poles of the system.

NOTES

Note 3.1 Numerical calculation of the transition matrix

It might seem that the numerical determination of the state-transition matrix

$$\Phi(T) = e^{AT}$$

with T fixed is a fairly routine numerical task. Algorithms can be based on the series definition

$$\Phi(T) = e^{AT} = I + AT + A^2T^2/2! + \cdots$$

or on the basic definition of an exponential

$$e^{AT} = \lim_{n\to\infty}(I + AT/n)^n$$

The transition matrix can also be computed by numerical integration of the matrix differential equation $\dot{\Phi} = A\Phi$ with the initial condition $\Phi(0) = I$. A variety of numerical integration algorithms (e.g., Runge-Kutta, predictor-corrector, implicit) and implemented computer codes are available.

It is also possible to transform A to Jordan canonical form (diagonal form for nonrepeated eigenvalues)

$$A = V\bar{A}V^{-1}$$

where $\bar{A}$ is in the Jordan form as given by (3.118). Then

$$e^{AT} = Ve^{\bar{A}T}V^{-1}$$

and $e^{\bar{A}T}$ has a particularly simple form. (When $\bar{A} = \Lambda = \text{diag}[s_1, s_2, \ldots, s_k]$, then $e^{\bar{A}T} = \text{diag}[e^{s_1T}, e^{s_2T}, \ldots, e^{s_kT}]$.) A number of algorithms are available for finding the eigenvalues of A (i.e., $s_1, \ldots, s_k$) and the corresponding transformation matrix V.

Notwithstanding the abundance of potentially suitable algorithms, when the dimension of A is large and when the eigenvalues have a range of several orders of magnitude, an accurate efficient algorithm for computing e^{AT} is not trivial.

Note 3.2 Time-varying systems

If we assume that the laws of nature do not change with time, we should not expect to encounter time-varying differential equations in the description of physical processes. Nonlinear, yes; but time-varying, no. Even if we accept this hypothesis, however, it is often necessary to deal with time-varying systems as an *approximate* representation of the physical world. Consider, for example, the motion of an aircraft, for which a set of time-invariant, but highly nonlinear equations can be written using established methods. These differential equations would be appropriate for use in an accurate simulation of the aircraft behavior. But for purposes of design it may be necessary to use a simplified, linear model. When the dynamics are linearized, the resulting differential equations (i.e., the A and B matrices) will have coefficients that depend on such variables as dynamic pressure $Q = \rho v^2/2$ which depend on time.

Example 3B is another example of how a nonlinear time-invariant system is approximated by a linear, time-varying system.

Note 3.3 Laplace transform of exponential

To show that the Laplace transform of e^{At} is $(sI - A)^{-1}$ consider the special case in which A is similar to a diagonal matrix $A = V\Lambda V^{-1}$ where $\Lambda = \text{diag}[s_1, s_2, \ldots, s_k]$. Then $e^{At} = V[e^{s_1 t}, \ldots, e^{s_k t}]V^{-1}$. Then the Laplace transform of e^{At} is $V[(s - s_1)^{-1}, \ldots, (s - s_k)^{-1}]V^{-1} = V(sI - \Lambda)^{-1}V^{-1} = (sI - A)^{-1}$. There are many other ways of showing this.

Note 3.4 Schuler period; inertial navigation

The period of a pendulum is $T = 2\pi\sqrt{l/g}$ (independent of the mass of the bob, which is why a pendulum clock can be extremely accurate). A pendulum having a length l equal to the earth's radius has a period of 84.4 minutes which is commonly called the Schuler period in honor of the German applied physicist Max Schuler,[7] who showed in 1923 that any pendulum having this length would remain vertical even if the pivot moves. This principle is the basis of inertial navigation systems. The orientation of the accelerometers in the system is kept constant by locating them on a "synthetic Schuler pendulum" in which the effect of the long pendulum arm is achieved by use of precise gyros.

Having become extremely sophisticated after World War II, inertial navigation technology is critical in strategic missiles and most military aircraft. It is also used extensively for navigation of modern transoceanic aircraft. Some of the analytical methods of inertial navigation may be found in [8].

Note 3.5 Minimal realizations

Several methods are displayed in Sec. 3.7 for realizing the transfer functions of a system with one input and l outputs, or with m inputs and one output, by a system of order k, where k is the degree of the characteristic polynomial of the system, i.e., the lowest common denominator of all the scalar transfer functions. By using several realizations in parallel it is possible to realize a system with m inputs and l outputs by a system of order $r = k \cdot \min(l, m)$. But it may be possible to realize the matrix of transfer functions by a system of order lower than r. For example, the system of transfer functions may have been obtained from a known system of differential equations of kth order as in Examples 2G or 2H. No matter how many inputs or outputs such a system may have, we know how to realize the transfer functions from all the inputs to all the outputs with a system of kth order.

If the transfer functions alone from the inputs to the outputs are given, however, the minimum number of differential equations (or integrators, in the block diagram representation) is not obvious, and the determination of this "minimum realization" is a significant and nontrivial problem. The problem is important not out of a desire to economize on hardware—a few integrators more or less is hardly of consequence—but because a nonminimum realization is either uncontrollable or unobservable (or both) in the sense defined and explained in Chap. 5, and may cause theoretical or computational difficulties.

The theory of minimum realizations is fundamental to the algebraic treatment of linear systems, as presented by Kailath,[4] for example. Unfortunately, this theory falls far outside the scope of the present text.

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