Slotine & Li (1991). Applied Nonlinear Control. Prentice Hall

Chapter 3
Fundamentals of Lyapunov Theory

Given a control system, the first and most important question about its various properties is whether it is stable, because an unstable control system is typically useless and potentially dangerous. Qualitatively, a system is described as stable if starting the system somewhere near its desired operating point implies that it will stay around the point ever after. The motions of a pendulum starting near its two equilibrium points, namely, the vertical up and down positions, are frequently used to illustrate unstable and stable behavior of a dynamic system. For aircraft control systems, a typical stability problem is intuitively related to the following question: will a trajectory perturbation due to a gust cause a significant deviation in the later flight trajectory? Here, the desired operating point of the system is the flight trajectory in the absence of disturbance. Every control system, whether linear or nonlinear, involves a stability problem which should be carefully studied.

The most useful and general approach for studying the stability of nonlinear control systems is the theory introduced in the late 19^{th} century by the Russian mathematician Alexandr Mikhailovich Lyapunov. Lyapunov's work, *The General Problem of Motion Stability*, includes two methods for stability analysis (the so-called linearization method and direct method) and was first published in 1892. The linearization method draws conclusions about a nonlinear system's local stability around an equilibrium point from the stability properties of its linear approximation. The direct method is not restricted to local motion, and determines the stability properties of a nonlinear system by constructing a scalar "energy-like" function for the system and examining the function's time variation. For over half a century, however

Lyapunov's pioneering work on stability received little attention outside Russia, although it was translated into French in 1908 (at the instigation of Poincare), and reprinted by Princeton University Press in 1947. The publication of the work of Lur'e and a book by La Salle and Lefschetz brought Lyapunov's work to the attention of the larger control engineering community in the early 1960's. Many refinements of Lyapunov's methods have since been developed. Today, Lyapunov's linearization method has come to represent the theoretical justification of linear control, while Lyapunov's direct method has become the most important tool for nonlinear system analysis and design. Together, the linearization method and the direct method constitute the so-called Lyapunov stability theory.

The objective of this and the next chapter is to present Lyapunov stability theory and illustrate its use in the analysis and the design of nonlinear systems. To prevent mathematical complexity from obscuring the theoretical concepts, this chapter presents the most basic results of Lyapunov theory in terms of autonomous (*i.e.*, time-invariant) systems, leaving more advanced topics to chapter 4. This chapter is organized as follows. In section 3.1, we provide some background definitions concerning nonlinear systems and equilibrium points. In section 3.2, various concepts of stability are described to characterize different aspects of system behavior. Lyapunov's linearization method is presented in section 3.3. The most useful theorems in the direct method are studied in section 3.4. Section 3.5 is devoted to the question of how to use these theorems to study the stability of particular classes of nonlinear systems. Section 3.6 sketches how the direct method can be used as a powerful way of designing controllers for nonlinear systems.

3.1 Nonlinear Systems and Equilibrium Points

Before addressing the main problems of defining and determining stability in the next sections, let us discuss some relatively simple background issues.

NONLINEAR SYSTEMS

A nonlinear dynamic system can usually be represented by a set of nonlinear differential equations in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{3.1}$$

where $\mathbf{f}$ is a $n \times 1$ nonlinear vector function, and $\mathbf{x}$ is the $n \times 1$ state vector. A particular value of the state vector is also called a point because it corresponds to a point in the state-space. The number of states n is called the *order* of the system. A solution $\mathbf{x}(t)$ of the equations (3.1) usually corresponds to a curve in state space as t varies from zero to infinity, as already seen in phase plane analysis for the case $n = 2$. This curve is generally referred to as a *state trajectory* or a *system trajectory*.

It is important to note that although equation (3.1) does not explicitly contain the control input as a variable, it is directly applicable to feedback control systems. The reason is that equation (3.1) can represent the *closed-loop* dynamics of a feedback control system, with the control input being a function of state **x** and time t, and therefore disappearing in the closed-loop dynamics. Specifically, if the plant dynamics is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

and some control law has been selected

$$\mathbf{u} = \mathbf{g}(\mathbf{x}, t)$$

then the closed-loop dynamics is

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}, \mathbf{g}(\mathbf{x}, t), t]$$

which can be rewritten in the form (3.1). Of course, equation (3.1) can also represent dynamic systems where no control signals are involved, such as a freely swinging pendulum.

A special class of nonlinear systems are *linear systems*. The dynamics of linear systems are of the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}$$

where A(t) is an $n \times n$ matrix.

AUTONOMOUS AND NON-AUTONOMOUS SYSTEMS

Linear systems are classified as either time-varying or time-invariant, depending on whether the system matrix **A** varies with time or not. In the more general context of nonlinear systems, these adjectives are traditionally replaced by "autonomous" and "non-autonomous".

Definition 3.1 *The nonlinear system (3.1) is said to be autonomous if* **f** *does not depend explicitly on time, i.e.*, if the system's state equation can be written

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{3.2}$$

Otherwise, the system is called non-autonomous .

Obviously, linear time-invariant (LTI) systems are autonomous and linear time-varying (LTV) systems are non-autonomous. The second-order systems studied in chapter 2 are all autonomous.

Strictly speaking, all physical systems are non-autonomous, because none of their dynamic characteristics is strictly time-invariant. The concept of an autonomous system is an idealized notion, like the concept of a linear system. In practice, however, system properties often change very slowly, and we can neglect their time variation without causing any practically meaningful error.

It is important to note that for control systems, the above definition is made on the *closed-loop dynamics*. Since a control system is composed of a controller and a plant (including sensor and actuator dynamics), the non-autonomous nature of a control system may be due to a time-variation either in the plant or in the control law. Specifically, a time-invariant plant with dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

may lead to a non-autonomous closed-loop system if a controller dependent on time t is chosen, *i.e.*, if $\mathbf{u} = \mathbf{g}(\mathbf{x}, t)$. For example, the closed-loop system of the simple plant $\dot{x} = -x + u$ can be nonlinear and non-autonomous by choosing u to be nonlinear and time-varying (*e.g.*, $u = -x^2 \sin t$). In fact, adaptive controllers for linear time-invariant plants usually make the closed-loop control systems nonlinear and non-autonomous.

The fundamental difference between autonomous and non-autonomous systems lies in the fact that the state trajectory of an autonomous system is independent of the initial time, while that of a non-autonomous system generally is not. As we will see in the next chapter, this difference requires us to consider the initial time explicitly in defining stability concepts for non-autonomous systems, and makes the analysis more difficult than that of autonomous systems.

It is well known that the analysis of linear time-invariant systems is much easier than that of linear time-varying systems. The same is true with nonlinear systems. Generally speaking, autonomous systems have relatively simpler properties and their analysis is much easier. For this reason, *in the remainder of this chapter, we will concentrate on the analysis of autonomous systems*, represented by (3.2). Extensions of the concepts and results to non-autonomous systems will be studied in chapter 4.

EQUILIBRIUM POINTS

It is possible for a system trajectory to correspond to only a single point. Such a point is called an equilibrium point. As we shall see later, many stability problems are naturally formulated with respect to equilibrium points.

Definition 3.2 *A state* $\mathbf{x}^*$ *is an equilibrium state (or equilibrium point) of the system if once* $\mathbf{x}(t)$ *is equal to* $\mathbf{x}^*$*, it remains equal to* $\mathbf{x}^*$ *for all future time.*

Mathematically, this means that the constant vector $\mathbf{x}^*$ satisfies

$$\mathbf{0} = \mathbf{f}(\mathbf{x}^*) \tag{3.3}$$

Equilibrium points can be found by solving the nonlinear algebraic equations (3.3).

A linear time-invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{3.4}$$

has a single equilibrium point (the origin $\mathbf{0}$) if $\mathbf{A}$ is nonsingular. If $\mathbf{A}$ is singular, it has an infinity of equilibrium points, which are contained in the null-space of the matrix $\mathbf{A}$, *i.e.*, the subspace defined by $\mathbf{A}\,\mathbf{x} = \mathbf{0}$. This implies that the equilibrium points are not isolated, as reflected by the example $\ddot{x} + \dot{x} = 0$, for which all points on the x axis of the phase plane are equilibrium points.

A nonlinear system can have several (or infinitely many) isolated equilibrium points, as seen in Example 1.1. The following example involves a familiar physical system.

Example 3.1: The Pendulum

Consider the pendulum of Figure 3.1, whose dynamics is given by the following nonlinear autonomous equation

$$MR^2\,\ddot{\theta} + b\,\dot{\theta} + MgR\sin\theta = 0 \tag{3.5}$$

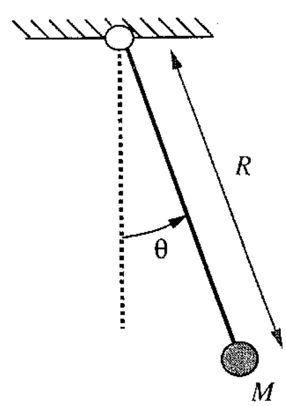

Figure 3.1 : The pendulum

where R is the pendulum's length, M its mass, b the friction coefficient at the hinge, and g the gravity constant. Letting $x_1 = \theta$, $x_2 = \dot{\theta}$, the corresponding state-space equation is

$$\dot{x}_1 = x_2 \tag{3.6a}$$

$$\dot{x}_2 = -\frac{b}{MR^2}x_2 - \frac{g}{R}\sin x_1 \tag{3.6b}$$

Therefore, the equilibrium points are given by

$$x_2 = 0 , \quad \sin x_1 = 0$$

which leads to the points (0 [2π], 0) and (π [2π], 0). Physically, these points correspond to the pendulum resting exactly at the vertical up and down positions. □

In linear system analysis and design, for notational and analytical simplicity, we often transform the linear system equations in such a way that the equilibrium point is the origin of the state-space. We can do the same thing for nonlinear systems (3.2), about a *specific* equilibrium point. Let us say that the equilibrium point of interest is $\mathbf{x}^*$. Then, by introducing a new variable

$$\mathbf{y} = \mathbf{x} - \mathbf{x}^*$$

and substituting $\mathbf{x} = \mathbf{y} + \mathbf{x}^*$ into equations (3.2), a new set of equations on the variable $\mathbf{y}$ are obtained

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y} + \mathbf{x}^*) \tag{3.7}$$

One can easily verify that there is a one-to-one correspondence between the solutions of (3.2) and those of (3.7), and that in addition, $\mathbf{y} = \mathbf{0}$, the solution corresponding to $\mathbf{x} = \mathbf{x}^*$, is an equilibrium point of (3.7). Therefore, instead of studying the behavior of the equation (3.2) in the neighborhood of $\mathbf{x}^*$, one can equivalently study the behavior of the equations (3.7) in the neighborhood of the origin.

NOMINAL MOTION

In some practical problems, we are not concerned with stability around an equilibrium point, but rather with the stability of a *motion*, *i.e*, whether a system will remain close to its original motion trajectory if slightly perturbed away from it, as exemplified by the aircraft trajectory control problem mentioned at the beginning of this chapter. We can show that this kind of motion stability problem can be transformed into an equivalent stability problem around an equilibrium point, although the equivalent system is now non-autonomous.

Let $\mathbf{x}^*(t)$ be the solution of equation (3.2), *i.e.*, the nominal motion trajectory, corresponding to initial condition $\mathbf{x}^*(0) = \mathbf{x}_o$. Let us now perturb the initial condition to be $\mathbf{x}(0) = \mathbf{x}_o + \delta\mathbf{x}_o$ and study the associated variation of the motion error

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$$

as illustrated in Figure 3.2. Since both $\mathbf{x}^*(t)$ and $\mathbf{x}(t)$ are solutions of (3.2), we have

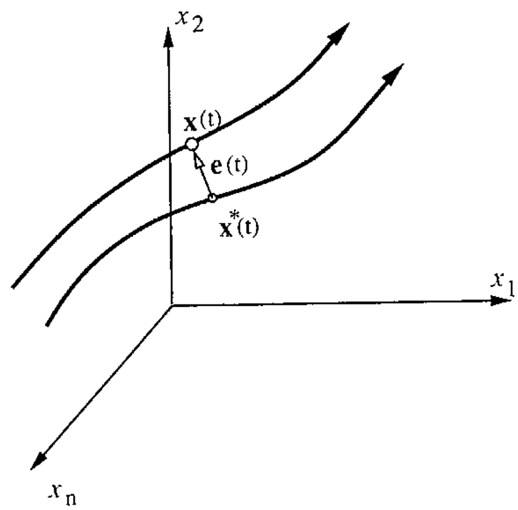

Figure 3.2 : Nominal and Perturbed Motions

$$\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*) \qquad \mathbf{x}(0) = \mathbf{x}_o$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \mathbf{x}(0) = \mathbf{x}_o + \delta\mathbf{x}_o$$

then $\mathbf{e}(t)$ satisfies the following non-autonomous differential equation

$$\dot{\mathbf{e}} = \mathbf{f}(\mathbf{x}^* + \mathbf{e}, t) - \mathbf{f}(\mathbf{x}^*, t) = \mathbf{g}(\mathbf{e}, t) \qquad (3.8)$$

with initial condition $\mathbf{e}(0) = \delta\mathbf{x}_o$. Since $\mathbf{g}(\mathbf{0}, t) = \mathbf{0}$, the new dynamic system, with $\mathbf{e}$ as state and $\mathbf{g}$ in place of $\mathbf{f}$, has an equilibrium point at the origin of the state space. Therefore, instead of studying the deviation of $\mathbf{x}(t)$ from $\mathbf{x}^*(t)$ for the original system, we may simply study the stability of the perturbation dynamics (3.8) with respect to the equilibrium point $\mathbf{0}$. Note, however, that the perturbation dynamics is non-autonomous, due to the presence of the nominal trajectory $\mathbf{x}^*(t)$ on the right-hand side. *Each* particular nominal motion of an autonomous system corresponds to an equivalent non-autonomous system, whose study requires the non-autonomous system analysis techniques to be presented in chapter 4.

Let us now illustrate this important transformation on a specific system.

Example 3.2: Consider the autonomous mass-spring system

$$m\ddot{x} + k_1 x + k_2 x^3 = 0$$

which contains a nonlinear term reflecting the hardening effect of the spring. Let us study the stability of the motion $x^*(t)$ which starts from initial position x_o.

Assume that we slightly perturb the initial position to be $x(0) = x_o + \delta x_o$. The resulting system trajectory is denoted as $x(t)$. Proceeding as before, the equivalent differential equation governing the motion error e is

$$m\ddot{e} + k_1 e + k_2 [\, e^3 + 3e^2 x^*(t) + 3e x^{*2}(t) \,] = 0$$

Clearly, this is a non-autonomous system. □

Of course, one can also show that for non-autonomous nonlinear systems, the stability problem around a nominal motion can also be transformed as a stability problem around the origin for an equivalent non-autonomous system.

Finally, note that if the original system is autonomous and *linear*, in the form (3.4), then the equivalent system is still autonomous, since it can be written

$$\dot{\mathbf{e}} = \mathbf{A}\,\mathbf{e}$$

3.2 Concepts of Stability

In the beginning of this chapter, we introduced the intuitive notion of stability as a kind of well-behavedness around a desired operating point. However, since nonlinear systems may have much more complex and exotic behavior than linear systems, the mere notion of stability is not enough to describe the essential features of their motion. A number of more refined stability concepts, such as asymptotic stability, exponential stability and global asymptotic stability, are needed. In this section, we define these stability concepts formally, for autonomous systems, and explain their practical meanings.

A few simplifying notations are defined at this point. Let $\mathbf{B}_R$ denote the spherical region (or ball) defined by $\|\mathbf{x}\| < R$ in state-space, and $\mathbf{S}_R$ the sphere itself, defined by $\|\mathbf{x}\| = R$.

STABILITY AND INSTABILITY

Let us first introduce the basic concepts of stability and instability.

Definition 3.3 *The equilibrium state* $\mathbf{x} = \mathbf{0}$ *is said to be* stable *if, for any* $R > 0$, *there exists* $r > 0$, *such that if* $\|\mathbf{x}(0)\| < r$, *then* $\|\mathbf{x}(t)\| < R$ *for all* $t \geq 0$. *Otherwise, the equilibrium point is* unstable.

Essentially, stability (also called *stability in the sense of Lyapunov*, or *Lyapunov stability*) means that the system trajectory can be kept arbitrarily close to the origin by starting sufficiently close to it. More formally, the definition states that the origin is stable, if, given that we do not want the state trajectory $\mathbf{x}(t)$ to get out of a ball of arbitrarily specified radius $\mathbf{B}_R$, a value $r(R)$ can be found such that starting the state from within the ball $\mathbf{B}_r$ at time 0 guarantees that the state will stay within the ball $\mathbf{B}_R$ thereafter. The geometrical implication of stability is indicated by curve 2 in Figure 3.3. Chapter 2 provides examples of stable equilibrium points in the case of second-order systems, such as the origin for the mass-spring system of Example 2.1, or stable nodes or foci in the local linearization of a nonlinear system.

Throughout the book, we shall use the standard mathematical abbreviation symbols:

$\forall$ to mean "for any"
$\exists$ for "there exists"
$\in$ for "in the set"
=> for "implies that"

Of course, we shall say interchangeably that A implies B, or that A is a *sufficient condition* of B, or that B is a *necessary condition* of A. If $A \Rightarrow B$ and $B \Rightarrow A$, then A and B are *equivalent*, which we shall denote by $A \Leftrightarrow B$.

Using these symbols, Definition 3.3 can be written

$$\forall R > 0 , \exists r > 0 , \ \| \mathbf{x}(0) \| < r \quad \Rightarrow \quad \forall t \geq 0 , \| \mathbf{x}(t) \| < R$$

or, equivalently

$$\forall R > 0 , \exists r > 0 , \ \mathbf{x}(0) \in \mathbf{B}_r \quad \Rightarrow \quad \forall t \geq 0 , \mathbf{x}(t) \in \mathbf{B}_R$$

Conversely, an equilibrium point is unstable if there exists at least *one* ball $\mathbf{B}_R$, such that for *every* $r > 0$, no matter how small, it is always possible for the system trajectory to start somewhere within the ball $\mathbf{B}_r$ and eventually leave the ball $\mathbf{B}_R$ (Figure 3.3). Unstable nodes or saddle points in second-order systems are examples of unstable equilibria. Instability of an equilibrium point is typically undesirable, because it often leads the system into limit cycles or results in damage to the involved mechanical or electrical components.

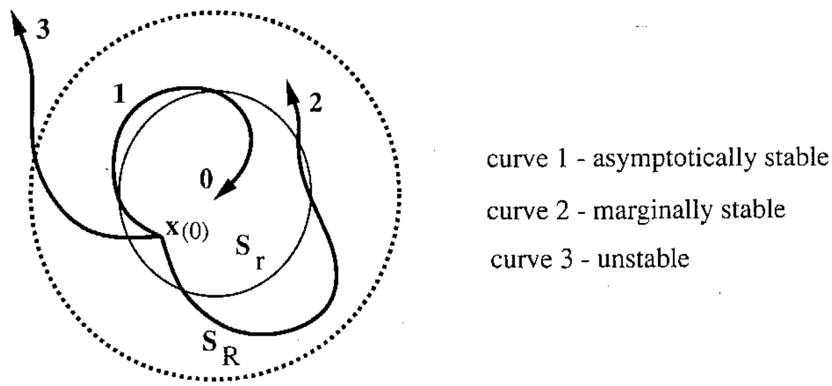

Figure 3.3 : Concepts of stability

It is important to point out the qualitative difference between instability and the intuitive notion of "blowing up" (all trajectories close to origin move further and further away to infinity). In linear systems, instability is equivalent to blowing up, because unstable poles always lead to exponential growth of the system states. However, for nonlinear systems, blowing up is only one way of instability. The following example illustrates this point.

Example 3.3: Instability of the Van der Pol Oscillator

The Van der Pol oscillator of Example 2.6 is described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2$$

One easily shows that the system has an equilibrium point at the origin.

As pointed out in section 2.2 and seen in the phase portrait of Figure 2.8, system trajectories starting from any non-zero initial states all asymptotically approach a limit cycle. This implies that, if we choose R in Definition 3.3 to be small enough for the circle of radius R to fall completely within the closed-curve of the limit cycle, then system trajectories starting near the origin will eventually get out of this circle (Figure 3.4). This implies instability of the origin.

Thus, even though the state of the system does remain around the equilibrium point in a certain sense, it cannot stay *arbitrarily* close to it. This is the fundamental distinction between stability and instability. □

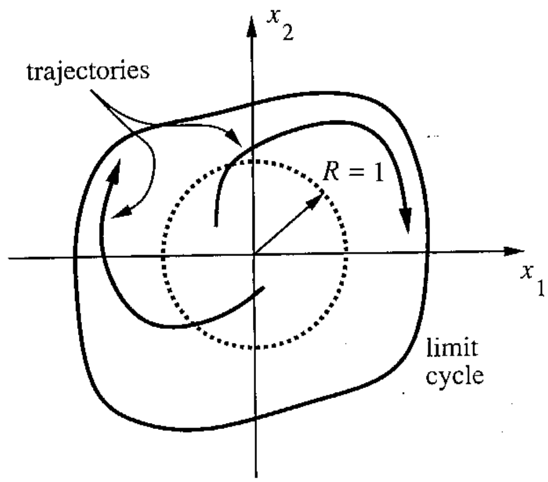

Figure 3.4 : Unstable origin of the Van der Pol Oscillator

ASYMPTOTIC STABILITY AND EXPONENTIAL STABILITY

In many engineering applications, Lyapunov stability is not enough. For example, when a satellite's attitude is disturbed from its nominal position, we not only want the satellite to maintain its attitude in a range determined by the magnitude of the disturbance, *i.e.*, Lyapunov stability, but also require that the attitude gradually go back to its original value. This type of engineering requirement is captured by the concept of asymptotic stability.

Definition 3.4 *An equilibrium point* **0** *is* *asymptotically stable* *if it is stable, and if in addition there exists some* $r > 0$ *such that* $\| \mathbf{x}(0) \| < r$ *implies that* $\mathbf{x}(t) \rightarrow \mathbf{0}$ *as* $t \rightarrow \infty$.

Asymptotic stability means that the equilibrium is stable, and that in addition, states started close to **0** actually converge to **0** as time t goes to infinity. Figure 3.3 shows that system trajectories starting from within the ball $\mathbf{B}_r$ converge to the origin. The ball $\mathbf{B}_r$ is called a *domain of attraction* of the equilibrium point (while *the* domain of attraction of the equilibrium point refers to the largest such region, *i.e.*, to the set of all points such that trajectories initiated at these points eventually converge to the origin). An equilibrium point which is Lyapunov stable but not asymptotically stable is called *marginally stable.*

One may question the need for the explicit stability requirement in the definition above, in view of the second condition of state convergence to the origin. However, it it easy to build counter-examples that show that state convergence does not necessarily imply stability. For instance, a simple system studied by Vinograd has trajectories of the form shown in Figure 3.5. All the trajectories starting from non-zero initial points within the unit disk first reach the curve **C** before converging to the origin. Thus, the origin is *unstable* in the sense of Lyapunov, despite the state convergence. Calling such a system unstable is quite reasonable, since a curve such as **C** may be outside the region where the model is valid – for instance, the subsonic and supersonic dynamics of a high-performance aircraft are radically different, while, with the problem under study using subsonic dynamic models, **C** could be in the supersonic range.

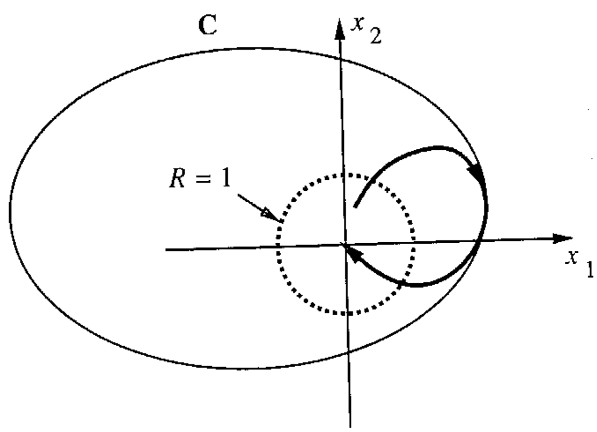

Figure 3.5 : State convergence does not imply stability

In many engineering applications, it is still not sufficient to know that a system will converge to the equilibrium point after infinite time. There is a need to estimate *how fast* the system trajectory approaches **0**. The concept of *exponential stability* can be used for this purpose.

Definition 3.5 *An equilibrium point* **0** *is* *exponentially stable* *if there exist two strictly positive numbers* α *and* λ *such that*

$$\forall\, t > 0, \quad \| \mathbf{x}(t) \| \leq \alpha \| \mathbf{x}(0) \| e^{-\lambda t} \qquad (3.9)$$

in some ball $\mathbf{B}_r$ *around the origin.*

In words, (3.9) means that the state vector of an exponentially stable system converges to the origin faster than an exponential function. The positive number λ is often called the *rate* of exponential convergence. For instance, the system

$$\dot{x} = -(1 + \sin^2 x)\, x$$

is exponentially convergent to $x = 0$ with a rate $\lambda = 1$. Indeed, its solution is

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$$x(t) = x(0)\exp\left(-\int_o^t [1+\sin^2(x(\tau))]\, d\tau\right)$$

and therefore

$$|x(t)| \leq |x(0)|\, e^{-t}$$

Note that exponential stability implies asymptotic stability. But asymptotic stability does not guarantee exponential stability, as can be seen from the system

$$\dot{x} = -x^2\,, \quad x(0) = 1 \tag{3.10}$$

whose solution is $x = 1/(1+t)$, a function slower than any exponential function $e^{-\lambda t}$ (with $\lambda > 0$).

The definition of exponential convergence provides an explicit bound on the state at any time, as seen in (3.9). By writing the positive constant α as $\alpha = e^{\lambda t_o}$, it is easy to see that, after a time of $\tau_o + (1/\lambda)$, the magnitude of the state vector decreases to less than 35% ($\approx e^{-1}$) of its original value, similarly to the notion of *time-constant* in a linear system. After $\tau_o + (3/\lambda)$, the state magnitude $\|\mathbf{x}(t)\|$ will be less than 5% ($\approx e^{-3}$) of $\|\mathbf{x}(0)\|$.

LOCAL AND GLOBAL STABILITY

The above definitions are formulated to characterize the *local* behavior of systems, *i.e.*, how the state evolves after starting near the equilibrium point. Local properties tell little about how the system will behave when the initial state is some distance away from the equilibrium, as seen for the nonlinear system in Example 1.1. Global concepts are required for this purpose.

Definition 3.6 *If asymptotic (or exponential) stability holds for any initial states, the equilibrium point is said to be asymptotically (or exponentially) stable* <u>*in the large*</u>*. It is also called* <u>*globally*</u> *asymptotically (or exponentially) stable.*

For instance, in Example 1.2 the linearized system is globally asymptotically stable, but the original system is not. The simple system in (3.10) is also globally asymptotically stable, as can be seen from its solutions.

Linear time-invariant systems are either asymptotically stable, or marginally stable, or unstable, as can be be seen from the modal decomposition of linear system solutions; linear asymptotic stability is always global and exponential, and linear instability always implies exponential blow-up. *This explains why the refined notions of stability introduced here were not previously encountered in the study of linear systems.* They are explicitly needed only for nonlinear systems.

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3 Linearization and Local Stability

Lyapunov's linearization method is concerned with the *local* stability of a nonlinear system. It is a formalization of the intuition that a nonlinear system should behave similarly to its linearized approximation for small range motions. Because all physical systems are inherently nonlinear, Lyapunov's linearization method serves as the fundamental *justification of using linear control techniques* in practice, *i.e.*, shows that stable design by linear control guarantees the stability of the original physical system locally.

Consider the autonomous system in (3.2), and assume that $\mathbf{f}(\mathbf{x})$ is continuously differentiable. Then the system dynamics can be written as

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} + \mathbf{f}_{h.o.t.}(\mathbf{x}) \tag{3.11}$$

where $\mathbf{f}_{h.o.t.}$ stands for higher-order terms in $\mathbf{x}$. Note that the above Taylor expansion starts directly with the first-order term, due to the fact that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, since $\mathbf{0}$ is an equilibrium point. Let us use the constant matrix $\mathbf{A}$ to denote the Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$ at $\mathbf{x} = \mathbf{0}$ (an $n \times n$ matrix of elements $\partial f_i / \partial x_j$)

$$\mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}}$$

Then, the system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{3.12}$$

is called the linearization (or linear approximation) of the original nonlinear system at the equilibrium point $\mathbf{0}$.

Note that, similarly, starting with a non-autonomous nonlinear system with a control input $\mathbf{u}$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

such that $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, we can write

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})} \mathbf{x} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})} \mathbf{u} + \mathbf{f}_{h.o.t.}(\mathbf{x}, \mathbf{u})$$

where $\mathbf{f}_{h.o.t.}$ stands for higher-order terms in $\mathbf{x}$ and $\mathbf{u}$. Letting $\mathbf{A}$ denote the Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$ at $(\mathbf{x} = \mathbf{0}, \mathbf{u} = \mathbf{0})$, and $\mathbf{B}$ denote the Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{u}$ at the same point (an $n \times m$ matrix of elements $\partial f_i / \partial u_j$, where m is the number of inputs)

$$\mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})} \qquad \mathbf{B} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})}$$

the system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$$

is the linearization (or linear approximation) of the original nonlinear system at $(\mathbf{x} = \mathbf{0}, \mathbf{u} = \mathbf{0})$.

Furthermore, the choice of a control law of the form $\mathbf{u} = \mathbf{u}(\mathbf{x})$ (with $\mathbf{u}(\mathbf{0}) = \mathbf{0}$) transforms the original non-autonomous system into an autonomous closed-loop system, having $\mathbf{x} = \mathbf{0}$ as an equilibrium point. Linearly approximating the control law as

$$\mathbf{u} \approx \left(\frac{d\mathbf{u}}{d\mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} = \mathbf{G}\,\mathbf{x}$$

the closed-loop dynamics can be linearly approximated as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \approx (\mathbf{A} + \mathbf{B}\,\mathbf{G})\,\mathbf{x}$$

Of course, the same linear approximation can be obtained by directly considering the autonomous closed-loop system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) = \mathbf{f}_1(\mathbf{x})$$

and linearizing the function $\mathbf{f}_1$ with respect to $\mathbf{x}$, at its equilibrium point $\mathbf{x} = \mathbf{0}$.

In practice, finding a system's linearization is often most easily done simply by neglecting any term of order higher than 1 in the dynamics, as we now illustrate.

Example 3.4: Consider the system

$$\dot{x}_1 = x_2^{\,2} + x_1 \cos x_2$$

$$\dot{x}_2 = x_2 + (x_1 + 1)\,x_1 + x_1 \sin x_2$$

Its linearized approximation about $\mathbf{x} = \mathbf{0}$ is

$$\dot{x}_1 \approx 0 + x_1 \cdot 1 = x_1$$

$$\dot{x}_2 \approx x_2 + 0 + x_1 + x_1\,x_2 \approx x_2 + x_1$$

The linearized system can thus be written

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}$$

= 0, u = 0).

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llustrate.

A similar procedure can be applied for a controlled system. Consider the system

$$\ddot{x} + 4\,\dot{x}^5 + (x^2+1)\,u = 0$$

The system can be linearly approximated about $\mathbf{x} = \mathbf{0}$ as

$$\ddot{x} + 0 + (0+1)\,u \approx 0$$

i.e., the linearized system can be written

$$\ddot{x} = -u$$

Assume that the control law for the original nonlinear system has been selected to be

$$u = \sin x + x^3 + \dot{x}\cos^2 x$$

then the linearized closed-loop dynamics is

$$\ddot{x} + \dot{x} + x = 0$$

□

The following result makes precise the relationship between the stability of the linear system (3.12) and that of the original nonlinear system (3.2).

Theorem 3.1 (Lyapunov's linearization method)

- *If the linearized system is strictly stable (i.e, if all eigenvalues of* **A** *are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system).*
- *If the linearized system is unstable (i.e, if at least one eigenvalue of* **A** *is strictly in the right-half complex plane), then the equilibrium point is unstable (for the nonlinear system).*
- *If the linearized system is marginally stable (i.e, all eigenvalues of* **A** *are in the left-half complex plane, but at least one of them is on the jω axis), then one cannot conclude anything from the linear approximation (the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system).*

While the proof of this theorem (which is actually based on Lyapunov's direct method, see Exercise 3.12) shall not be detailed, let us remark that its results are *intuitive*. A summary of the theorem is that it is true *by continuity*. If the linearized system is strictly stable, or strictly unstable, then, since the approximation is valid "not too far" from the equilibrium, the nonlinear system itself is locally stable, or locally unstable. However, if the linearized system is marginally stable, the higher-order terms in (3.11) can have a decisive effect on whether the nonlinear system is stable or unstable. As we shall see in the next section, simple nonlinear systems may be globally asymptotically stable while their linear approximations are only marginally stable: one simply cannot infer any stability property of a nonlinear system from its marginally stable linear approximation.

Example 3.5: As expected, it can be shown easily that the equilibrium points $(\theta = \pi\ [2\pi]\ ,\ \dot{\theta} = 0)$ of the pendulum of Example 3.1 are unstable. Consider for instance the equilibrium point $(\theta = \pi\ ,\ \dot{\theta} = 0)$. Since, in a neighborhood of $\theta = \pi$, we can write

$$\sin\theta \;=\; \sin\pi + \cos\pi\,(\theta - \pi) + h.o.t. \;=\; (\pi - \theta) + h.o.t.$$

thus, letting $\tilde{\theta} = \theta - \pi$, the system's linearization *about the equilibrium point* $(\theta = \pi\ ,\ \dot{\theta} = 0)$ is

$$\ddot{\tilde{\theta}} + \frac{b}{MR^2}\,\dot{\tilde{\theta}} - \frac{g}{R}\,\tilde{\theta} = 0$$

Hence the linear approximation is unstable, and therefore so is the nonlinear system at this equilibrium point. □

Example 3.6: Consider the first order system

$$\dot{x} = a x + b x^5$$

The origin 0 is one of the two equilibrium points of this system. The linearization of this system around the origin is

$$\dot{x} = a x$$

The application of Lyapunov's linearization method indicates the following stability properties of the nonlinear system

- $a < 0$: asymptotically stable;
- $a > 0$: unstable;
- $a = 0$: cannot tell from linearization.

In the third case, the nonlinear system is

$$\dot{x} = b x^5$$

The linearization method fails while, as we shall see, the direct method to be described can easily solve this problem. □

Lyapunov's linearization theorem shows that linear control design is a matter of *consistency*: one must design a controller such that the system remain in its "linear range". It also stresses major limitations of linear design: how large is the linear range? What is the *extent* of stability (how large is r in Definition 3.3) ? These questions motivate a deeper approach to the nonlinear control problem, Lyapunov's direct method.

Lyapunov's Direct Method

The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation: if the total *energy* of a mechanical (or electrical) system is continuously dissipated, then the system, *whether linear or nonlinear*, must eventually settle down to an equilibrium point. Thus, we may conclude the stability of a system by examining the variation of a single *scalar* function.

Specifically, let us consider the nonlinear mass-damper-spring system in Figure 3.6, whose dynamic equation is

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_o x + k_1 x^3 = 0 \tag{3.13}$$

with $b\dot{x}|\dot{x}|$ representing nonlinear dissipation or damping, and $(k_o x + k_1 x^3)$ representing a nonlinear spring term. Assume that the mass is pulled away from the natural length of the spring by a large distance, and then released. Will the resulting motion be stable? It is very difficult to answer this question using the definitions of stability, because the general solution of this nonlinear equation is unavailable. The linearization method cannot be used either because the motion starts outside the linear range (and in any case the system's linear approximation is only marginally stable). However, examination of the system energy can tell us a lot about the motion pattern.

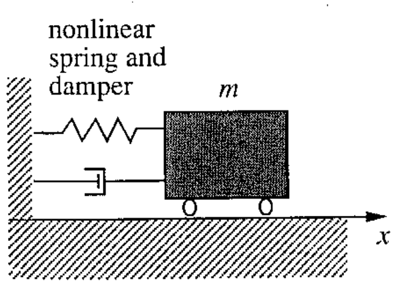

Figure 3.6 : A nonlinear mass-damper-spring system

The total mechanical energy of the system is the sum of its kinetic energy and its potential energy

$$V(\mathbf{x}) = \frac{1}{2} m\dot{x}^2 + \int_o^x (k_o\, x + k_1 x^3)\, dx \; = \; \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_o\, x^2 + \frac{1}{4} k_1 x^4 \qquad (3.14)$$

Comparing the definitions of stability and mechanical energy, one can easily see some relations between the mechanical energy and the stability concepts described earlier:

- zero energy corresponds to the equilibrium point ($\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$)
- asymptotic stability implies the convergence of mechanical energy to zero
- instability is related to the growth of mechanical energy

These relations indicate that the value of a scalar quantity, the mechanical energy, indirectly reflects the magnitude of the state vector; and furthermore, that the stability properties of the system can be characterized by the variation of the mechanical energy of the system.

The rate of energy variation during the system's motion is obtained easily by differentiating the first equality in (3.14) and using (3.13)

$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_o x + k_1\, x^3)\, \dot{x} = \; \dot{x}\, (-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3 \qquad (3.15)$$

Equation (3.15) implies that the energy of the system, starting from some initial value, is continuously dissipated by the damper until the mass settles down, *i.e.*, until $\dot{x} = 0$. Physically, it is easy to see that the mass must finally settle down at the natural length of the spring, because it is subjected to a non-zero spring force at any position other than the natural length.

The direct method of Lyapunov is based on a generalization of the concepts in the above mass-spring-damper system to more complex systems. Faced with a set of nonlinear differential equations, the basic procedure of Lyapunov's direct method is to generate a scalar "energy-like" function for the dynamic system, and examine the time variation of that scalar function. In this way, conclusions may be drawn on the stability of the set of differential equations without using the difficult stability definitions or requiring explicit knowledge of solutions.

3.4.1 Positive Definite Functions and Lyapunov Functions

The energy function in (3.14) has two properties. The first is a property of the function itself: it is strictly positive unless both state variables x and $\dot{x}$ are zero. The second property is a property associated with the dynamics (3.13): the function is monotonically decreasing when the variables x and $\dot{x}$ vary according to (3.13). In Lyapunov's direct method, the first property is formalized by the notion of *positive definite functions*, and the second is formalized by the so-called Lyapunov functions.

Let us discuss positive definite functions first.

Definition 3.7 *A scalar continuous function* $V(\mathbf{x})$ *is said to be* <u>*locally positive definite*</u> *if* $V(\mathbf{0}) = 0$ *and, in a ball* $\mathbf{B}_{R_o}$

$$\mathbf{x} \neq \mathbf{0} \quad => \quad V(\mathbf{x}) > 0$$

If $V(\mathbf{0}) = 0$ *and the above property holds over the whole state space, then* $V(\mathbf{x})$ *is said to be* <u>*globally positive definite*</u>.

For instance, the function

$$V(\mathbf{x}) = \frac{1}{2} M R^2 x_2{}^2 + MR(1 - \cos x_1)$$

which is the mechanical energy of the pendulum of Example 3.1, is locally positive definite. The mechanical energy (3.14) of the nonlinear mass-damper-spring system is globally positive definite. Note that, for that system, the kinetic energy $(1/2)\, m \dot{x}^2$ is *not* positive definite by itself, because it can equal zero for non-zero values of x.

The above definition implies that the function V has a unique minimum at the origin **0**. Actually, given any function having a *unique* minimum in a certain ball, we can construct a locally positive definite function simply by adding a constant to that function. For example, the function $V(\mathbf{x}) = x_1{}^2 + x_2{}^2 - 1$ is a lower bounded function with a unique minimum at the origin, and the addition of the constant 1 to it makes it a positive definite function. Of course, the function shifted by a constant has the same time-derivative as the original function.

Let us describe the geometrical meaning of locally positive definite functions. Consider a positive definite function $V(\mathbf{x})$ of two state variables x_1 and x_2. Plotted in a 3-dimensional space, $V(\mathbf{x})$ typically corresponds to a surface looking like an upward cup (Figure 3.7). The lowest point of the cup is located at the origin.

A second geometrical representation can be made as follows. Taking x_1 and x_2 as Cartesian coordinates, the level curves $V(x_1, x_2) = V_\alpha$ typically represent a set of ovals surrounding the origin, with each oval corresponding to a positive value of V_α. These ovals, often called *contour curves*, may be thought as the sections of the cup by horizontal planes, projected on the $(x_1\,, x_2)$ plane (Figure 3.8). Note that the contour curves do not intersect, because $V(x_1, x_2)$ is uniquely defined given $(x_1\,, x_2)$.

A few related concepts can be defined similarly, in a local or global sense, *i.e.*, a function $V(\mathbf{x})$ is *negative definite* if $-V(\mathbf{x})$ is positive definite; $V(\mathbf{x})$ is *positive semi-definite* if $V(0) = 0$ and $V(\mathbf{x}) \geq 0$ for $\mathbf{x} \neq \mathbf{0}$; $V(\mathbf{x})$ is *negative semi-definite* if $-V(\mathbf{x})$ is positive semi-definite. The prefix "semi" is used to reflect the possibility of V being

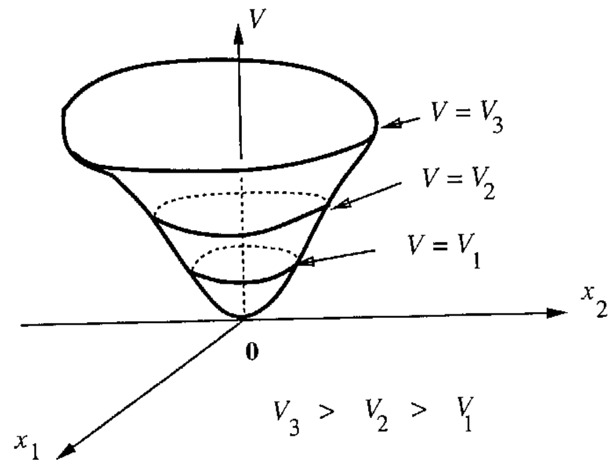

Figure 3.7 : Typical shape of a positive definite function $V(x_1, x_2)$

equal to zero for $\mathbf{x} \neq \mathbf{0}$. These concepts can be given geometrical meanings similar to the ones given for positive definite functions.

With $\mathbf{x}$ denoting the state of the system (3.2), a scalar function $V(\mathbf{x})$ actually represents an implicit function of time t. Assuming that $V(\mathbf{x})$ is differentiable, its derivative with respect to time can be found by the chain rule,

$$\dot{V} = \frac{dV(\mathbf{x})}{dt} = \frac{\partial V}{\partial \mathbf{x}}\,\dot{\mathbf{x}} = \frac{\partial V}{\partial \mathbf{x}}\,\mathbf{f}(\mathbf{x})$$

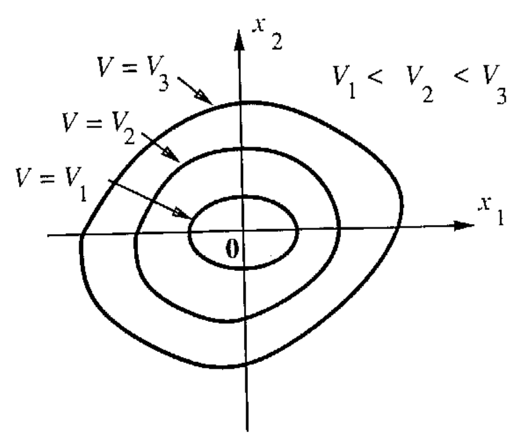

Figure 3.8 : Interpreting positive definite functions using contour curves

We see that, because **x** is required to satisfy the autonomous state equations (3.2), $\dot{V}$ only depends on **x**. It is often referred to as "the derivative of V along the system trajectory". For the system (3.13), $\dot{V}(\mathbf{x})$ is computed in (3.15) and found to be negative. Functions such as V in that example are given a special name because of their importance in Lyapunov's direct method.

Definition 3.8 *If, in a ball* $\mathbf{B}_{R_o}$ *, the function* $V(\mathbf{x})$ *is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system (3.2) is negative semi-definite, i.e.,*

$$\dot{V}(\mathbf{x}) \leq 0$$

then $V(\mathbf{x})$ *is said to be a* <u>*Lyapunov function*</u> *for the system (3.2).*

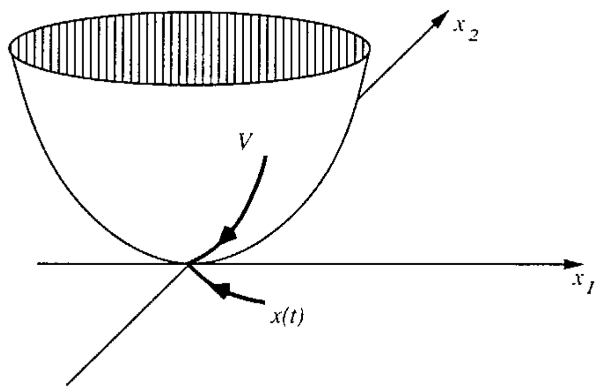

Figure 3.9 : Illustrating Definition 3.8 for $n = 2$

A Lyapunov function can be given simple geometrical interpretations. In Figure 3.9, the point denoting the value of $V(x_1, x_2)$ is seen to always point down an inverted cup. In Figure 3.10, the state point is seen to move across contour curves corresponding to lower and lower values of V.

3.4.2 Equilibrium Point Theorems

The relations between Lyapunov functions and the stability of systems are made precise in a number of theorems in Lyapunov's direct method. Such theorems usually have local and global versions. The local versions are concerned with stability properties in the neighborhood of equilibrium point and usually involve a locally positive definite function.

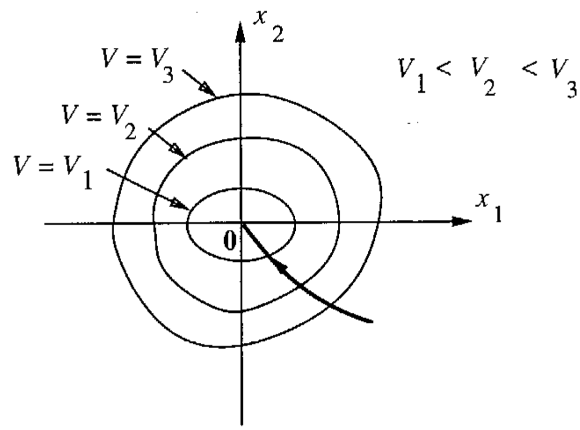

Figure 3.10 : Illustrating Definition 3.8 for $n = 2$ using contour curves

LYAPUNOV THEOREM FOR LOCAL STABILITY

Theorem 3.2 (Local Stability) *If, in a ball* $\mathbf{B}_{R_o}$, *there exists a scalar function* $V(\mathbf{x})$ *with continuous first partial derivatives such that*

- *$V(\mathbf{x})$ is positive definite (locally in* $\mathbf{B}_{R_o}$ *)*
- *$\dot{V}(\mathbf{x})$ is negative semi-definite (locally in* $\mathbf{B}_{R_o}$ *)*

then the equilibrium point $\mathbf{0}$ *is stable. If, actually, the derivative* $\dot{V}(\mathbf{x})$ *is locally negative definite in* $\mathbf{B}_{R_o}$, *then the stability is asymptotic.*

The proof of this fundamental result is conceptually simple, and is typical of many proofs in Lyapunov theory.

Proof: Let us derive the result using the geometric interpretation of a Lyapunov function, as illustrated in Figure 3.9 in the case $n = 2$. To show stability, we must show that given any strictly positive number R, there exists a (smaller) strictly positive number r such that any trajectory starting inside the ball $\mathbf{B}_r$ remains inside the ball $\mathbf{B}_R$ for all future time. Let m be the *minimum of V on the sphere* $\mathbf{S}_R$. Since V is continuous and positive definite, m exists and is strictly positive. Furthermore, since $V(\mathbf{0}) = 0$, there exists a ball $\mathbf{B}_r$ around the origin such that $V(\mathbf{x}) < m$ for any $\mathbf{x}$ inside the ball (Figure 3.11a). Consider now a trajectory whose initial point $\mathbf{x}(0)$ is within the ball $\mathbf{B}_r$. Since V is non-increasing along system trajectories, *V remains strictly smaller than m*, and therefore the trajectory cannot possibly cross the outside sphere $\mathbf{S}_R$. Thus, any trajectory starting inside the ball $\mathbf{B}_r$ remains inside the ball $\mathbf{B}_R$, and therefore Lyapunov stability is guaranteed.

Let us now assume that $\dot{V}$ is negative definite, and show asymptotic stability, by contradiction. Consider a trajectory starting in some ball $\mathbf{B}_r$ as constructed above (*e.g.*, the ball $\mathbf{B}_r$

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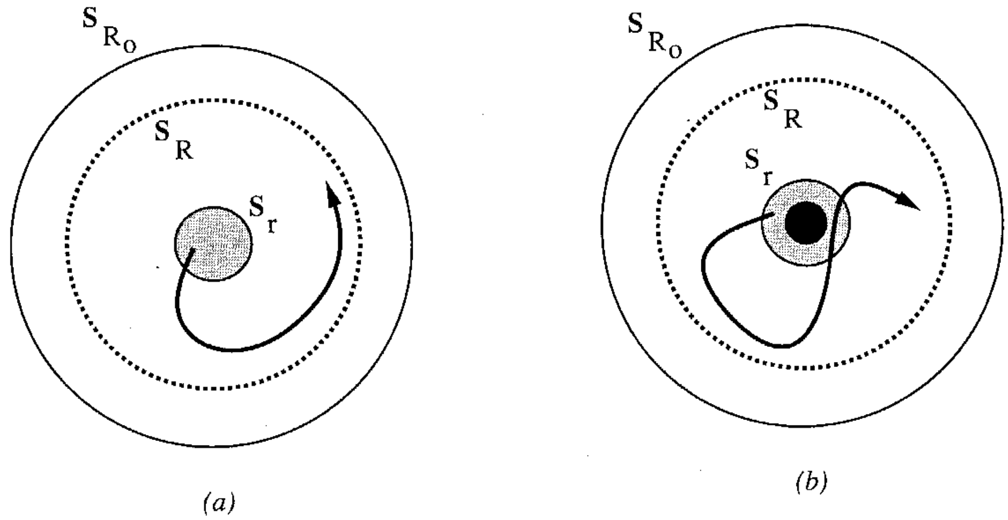

Figure 3.11 : Illustrating the proof of Theorem 3.2 for $n = 2$

corresponding to $R = R_o$). Then the trajectory will remain in the ball $\mathbf{B}_R$ for all future time. Since V is lower bounded and decreases continually, V tends towards a limit L, such that $\forall\, t \geq 0$, $V(\mathbf{x}(t)) \geq L$. Assume that this limit is not zero, *i.e.*, that $L > 0$. Then, since V is continuous and $V(\mathbf{0}) = 0$, there exists a ball $\mathbf{B}_{r_o}$ that the system trajectory never enters (Figure 3.11b). But then, *since* $-\dot{V}$ *is also continuous and positive definite*, and since $\mathbf{B}_R$ is bounded, $-\dot{V}$ must remain larger than some strictly positive number L_1. This is a contradiction, because it would imply that $V(t)$ decreases from its initial value V_o to a value strictly smaller than L, in a finite time smaller than $[V_o - L]/L_1$. Hence, all trajectories starting in $\mathbf{B}_r$ asymptotically converge to the origin. □

In applying the above theorem for analysis of a nonlinear system, one goes through the two steps of choosing a positive definite function, and then determining its derivative along the path of the nonlinear systems. The following example illustrates this procedure.

Example 3.7: Local Stability

A simple pendulum with viscous damping is described by

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

Consider the following scalar function

$$V(\mathbf{x}) \;=\; (1 - \cos\theta) + \frac{\dot{\theta}^2}{2}$$

One easily verifies that this function is locally positive definite. As a matter of fact, this function represents the total energy of the pendulum, composed of the sum of the potential energy and the kinetic energy. Its time-derivative is easily found to be

$$\dot{V}(\mathbf{x}) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = -\dot{\theta}^2 \le 0$$

Therefore, by invoking the above theorem, one concludes that the origin is a stable equilibrium point. In fact, using physical insight, one easily sees the reason why $\dot{V}(\mathbf{x}) \le 0$, namely that the damping term absorbs energy. Actually, $\dot{V}$ is precisely the power dissipated in the pendulum. However, with this Lyapunov function, one cannot draw conclusions on the asymptotic stability of the system, because $\dot{V}(\mathbf{x})$ is only negative semi-definite. □

The following example illustrates the asymptotic stability result.

Example 3.8: Asymptotic stability

Let us study the stability of the nonlinear system defined by

$$\dot{x}_1 = x_1\,(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$

$$\dot{x}_2 = 4x_1^2x_2 + x_2\,(x_1^2 + x_2^2 - 2)$$

around its equilibrium point at the origin. Given the positive definite function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

its derivative $\dot{V}$ along any system trajectory is

$$\dot{V} = 2\,(x_1^2 + x_2^2)\,(x_1^2 + x_2^2 - 2\,)$$

Thus, $\dot{V}$ is locally negative definite in the 2-dimensional ball $\mathbf{B}_2$, *i.e.*, in the region defined by $x_1^2 + x_2^2 < 2$. Therefore, the above theorem indicates that the origin is asymptotically stable. □

LYAPUNOV THEOREM FOR GLOBAL STABILITY

The above theorem applies to the local analysis of stability. In order to assert *global asymptotic stability* of a system, one might naturally expect that the ball $\mathbf{B}_{R_o}$ in the above local theorem has to be expanded to be the whole state-space. This is indeed necessary, but it is not enough. An additional condition on the function V has to be satisfied: $V(\mathbf{x})$ must be *radially unbounded*, by which we mean that $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$ (in other words, as $\mathbf{x}$ tends to infinity in *any* direction). We then obtain the following powerful result:

Chap. 3

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Theorem 3.3 (Global Stability) *Assume that there exists a scalar function V of the state* $\mathbf{x}$, *with continuous first order derivatives such that*

- $V(\mathbf{x})$ *is positive definite*
- $\dot{V}(\mathbf{x})$ *is negative definite*
- $V(\mathbf{x}) \to \infty$ *as* $\|\mathbf{x}\| \to \infty$

then the equilibrium at the origin is globally asymptotically stable.

Proof: The proof is the same as in the local case, by noticing that the radial unboundedness of V, combined with the negative definiteness of $\dot{V}$, implies that, given any initial condition $\mathbf{x}_o$, the trajectories remain in the *bounded* region defined by $V(\mathbf{x}) \leq V(\mathbf{x}_o)$. □

The reason for the radial unboundedness condition is to assure that the contour curves (or contour surfaces in the case of higher order systems) $V(\mathbf{x}) = V_\alpha$ correspond to closed curves. If the curves are not closed, it is possible for the state trajectories to drift away from the equilibrium point, even though the state keeps going through contours corresponding to smaller and smaller V_α's. For example, for the positive definite function $V = [\,x_1^2/(1+x_1^2)\,] + x_2^2$, the curves $V(\mathbf{x}) = V_\alpha$ for $V_\alpha > 1$ are open curves. Figure 3.12 shows the divergence of the state while moving toward lower and lower "energy" curves. Exercise 3.4 further illustrates this point on a specific system.

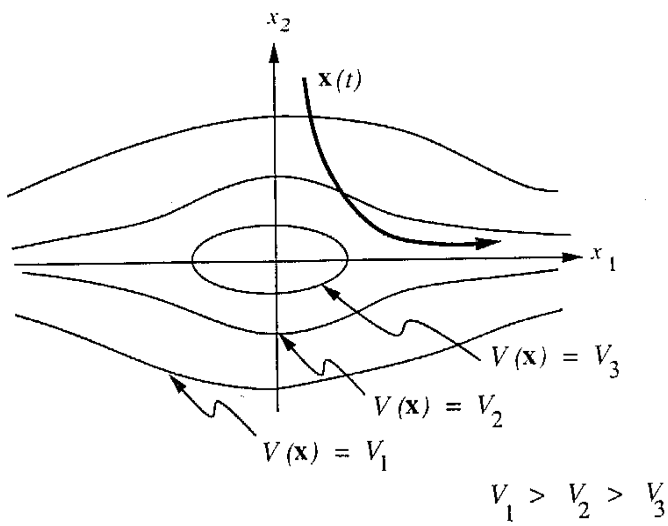

Figure 3.12 : Motivation of the radial unboundedness condition

Example 3.9: A class of first-order systems

Consider the nonlinear system

$$\dot{x} + c(x) = 0$$

where c is any continuous function of the same sign as its scalar argument x, *i.e.*,

$$x\,c(x) > 0 \quad \text{for} \quad x \neq 0$$

Intuitively, this condition indicates that $-c(x)$ "pushes" the system back towards its rest position $x = 0$, but is otherwise arbitrary. Since c is continuous, it also implies that $c(0) = 0$ (Figure 3.13).

Consider as the Lyapunov function candidate the square of the distance to the origin

$$V = x^2$$

The function V is radially unbounded, since it tends to infinity as $|x| \to \infty$. Its derivative is

$$\dot{V} = 2\,x\,\dot{x} = = -2\,x\,c(x)$$

Thus $\dot{V} < 0$ as long as $x \neq 0$, so that $x = 0$ is a globally asymptotically stable equilibrium point.

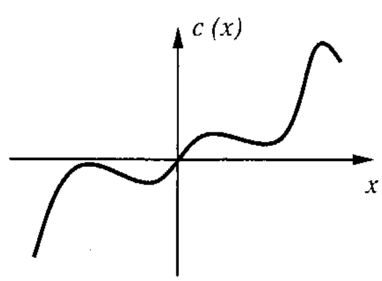

Figure 3.13 : The function $c(x)$

For instance, the system

$$\dot{x} = \sin^2 x - x$$

is globally asymptotically convergent to $x = 0$, since for $x \neq 0$, $\sin^2 x \leq |\sin x| < |x|$. Similarly, the system

$$\dot{x} = -x^3$$

is globally asymptotically convergent to $x = 0$. Notice that while this system's linear approximation ($\dot{x} \approx 0$) is inconclusive, even about local stability, the actual nonlinear system enjoys a strong stability property (*global asymptotic* stability). □

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Example 3.10: Consider the system

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

The origin of the state-space is an equilibrium point for this system. Let V be the positive definite function

$$V(\mathbf{x}) = x_1^2 + x_2^2$$

The derivative of V along any system trajectory is

$$\dot{V}(\mathbf{x}) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2)^2$$

which is negative definite. Therefore, the origin is a globally asymptotically stable equilibrium point. Note that the globalness of this stability result also implies that the origin is the *only* equilibrium point of the system. □

REMARKS

Many Lyapunov functions may exist for the same system. For instance, if V is a Lyapunov function for a given system, so is

$$V_1 = \rho V^{\alpha}$$

where ρ is any strictly positive constant and α is any scalar (not necessarily an integer) larger than 1. Indeed, the positive-definiteness of V implies that of V_1, the positive-definiteness (or positive semi-definiteness) of $-\dot{V}$ implies that of $-\dot{V}_1$, and (the radial unboundedness of V (if applicable) implies that of V_1.

More importantly, for a given system, specific choices of Lyapunov functions may yield more precise results than others. Consider again the pendulum of Example 3.7. The function

$$V(\mathbf{x}) = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}(\dot{\theta} + \theta)^2 + 2(1 - \cos\theta)$$

is also a Lyapunov function for the system, because locally

$$\dot{V}(\mathbf{x}) = -(\dot{\theta}^2 + \theta\sin\theta) \le 0$$

However, it is interesting to note that $\dot{V}$ is actually locally *negative definite*, and therefore, this modified choice of V, without obvious physical meaning, allows the asymptotic stability of the pendulum to be shown.

$$\ddot{x} + |x^2 - 1|\,\dot{x}^3 + x = \sin\frac{\pi x}{2}$$

For this system, we study, similarly to Example 3.14, the Lyapunov function

$$V = \frac{1}{2}\dot{x}^2 + \int_0^x (y - \sin\frac{\pi y}{2})\, dy$$

This function has two minima, at $x = \pm 1$; $\dot{x} = 0$, and a local *maximum* in x (a saddle point in the state-space) at $x = 0$; $\dot{x} = 0$. As in Example 3.14, the time-derivative of V is (without calculations)

$$\dot{V} = -\ |x^2 - 1|\,\dot{x}^4$$

i.e, the virtual power "dissipated" by the system. Now

$$\dot{V} = 0 \quad \Rightarrow \quad \dot{x} = 0 \ \text{ or } \ x = \pm 1$$

Let us consider each of these cases:

$$\dot{x} = 0 \quad \Rightarrow \quad \ddot{x} = \sin\frac{\pi x}{2} - x \neq 0 \ \text{ except if } \ x = 0 \ \text{ or } \ x = \pm 1$$

$$x = \pm 1 \quad \Rightarrow \quad \ddot{x} = 0$$

Thus, the invariant set theorem indicates that the system converges globally to ($x = 1$; $\dot{x} = 0$) or ($x = -1$; $\dot{x} = 0$), or to ($x = 0$; $\dot{x} = 0$). The first two of these equilibrium points are stable, since they correspond to local mimina of V (note again that linearization is inconclusive about their stability). By contrast, the equilibrium point ($x = 0$; $\dot{x} = 0$) is unstable, as can be shown from linearization ($\ddot{x} \approx (\pi/2 - 1)\, x$), or simply by noticing that because that point is a local maximum of V along the x axis, any small deviation in the x direction will drive the trajectory away from it. □

As noticed earlier, several Lyapunov functions may exist for a given system and therefore several associated invariant sets may be derived. The system then converges to the (necessarily non-empty) *intersection* of the invariant sets $\mathbf{M}_i$, which may give a more precise result than that obtained from any of the Lyapunov functions taken separately. Equivalently, one can notice that the sum of two Lyapunov functions for a given system is also a Lyapunov function, whose set $\mathbf{R}$ is the intersection of the individual sets $\mathbf{R}_i$.

3.5 System Analysis Based on Lyapunov's Direct Method

With so many theorems and so many examples presented in the last section, one may feel confident enough to attack practical nonlinear control problems. However, the theorems all make a basic assumption: an explicit Lyapunov function is somehow known. The question is therefore how to find a Lyapunov function for a specific problem. Yet, there is no general way of finding Lyapunov functions for nonlinear systems. This is a fundamental drawback of the direct method. Therefore, faced with specific systems, one has to use experience, intuition, and physical insights to search for an appropriate Lyapunov function. In this section, we discuss a number of techniques which can facilitate the otherwise blind search of Lyapunov functions.

We first show that, not surprisingly, Lyapunov functions can be systematically found to describe stable *linear* systems. Next, we discuss two of many mathematical methods that may be used to help finding a Lyapunov function for a given nonlinear system. We then consider the use of physical insights, which, when applicable, represents by far the most powerful and elegant way of approaching the problem, and is closest in spirit to the original intuition underlying the direct method. Finally, we discuss the use of Lyapunov functions in transient performance analysis.

3.5.1 Lyapunov Analysis of Linear Time-Invariant Systems

Stability analysis for linear time-invariant systems is well known. It is interesting, however, to develop Lyapunov functions for such systems. First, this allows us to describe both linear and nonlinear systems using *a common language*, allowing shared insights between the two classes. Second, as we shall detail later on, Lyapunov functions are "additive", like energy. In other words, Lyapunov functions for combinations of subsystems may be derived by adding the Lyapunov functions of the subsystems. Since nonlinear control systems may include linear components (whether in plant or in controller), we should be able to describe linear systems in the Lyapunov formalism.

We first review some basic results on matrix algebra, since the development of Lyapunov functions for linear systems will make extensive use of quadratic forms.

SYMMETRIC, SKEW-SYMMETRIC, AND POSITIVE DEFINITE MATRICES

Definition 3.10 *A square matrix* $\mathbf{M}$ *is symmetric if* $\mathbf{M} = \mathbf{M}^T$ *(in other words, if* $\forall\, i, j \;\; M_{ij} = M_{ji}$ *). A square matrix* $\mathbf{M}$ *is skew-symmetric if* $\mathbf{M} = -\mathbf{M}^T$ *(i.e, if* $\forall\, i, j \;\; M_{ij} = -M_{ji}$ *).*

An interesting fact is that any square $n \times n$ matrix $\mathbf{M}$ can be represented as the sum of a symmetric matrix and a skew-symmetric matrix. This can be shown by the following decomposition

$$\mathbf{M} = \frac{\mathbf{M}+\mathbf{M}^T}{2} + \frac{\mathbf{M}-\mathbf{M}^T}{2}$$

where the first term on the left side is symmetric and the second term is skew-symmetric.

Another interesting fact is that the quadratic function associated with a skew-symmetric matrix is always zero. Specifically, let $\mathbf{M}$ be a $n \times n$ skew-symmetric matrix and $\mathbf{x}$ an arbitrary $n \times 1$ vector. Then the definition of a skew-symmetric matrix implies that

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = -\,\mathbf{x}^T\mathbf{M}^T\mathbf{x}$$

Since $\mathbf{x}^T\mathbf{M}^T\mathbf{x}$ is a scalar, the left-hand side of the above equation can be replaced by its transpose. Therefore,

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = -\,\mathbf{x}^T\mathbf{M}\mathbf{x}$$

This shows that

$$\forall\, \mathbf{x}\,,\ \mathbf{x}^T\mathbf{M}\mathbf{x} = 0 \qquad (3.16)$$

In designing some tracking control systems for robots, for instance, this fact is very useful because it can simplify the control law, as we shall see in chapter 9.

Actually, property (3.16) is a *necessary and sufficient* condition for a matrix $\mathbf{M}$ to be skew-symmetric. This can be easily seen by applying (3.16) to the basis vectors $\mathbf{e}_i$:

$$[\ \forall\, i\,,\ \mathbf{e}_i^T\mathbf{M}_s\mathbf{e}_i = 0\ ] \ \ => \ \ [\ \forall\, i,\ M_{ii} = 0\ ]$$

and

$$[\ \forall\,(i,j)\,,\ (\mathbf{e}_i+\mathbf{e}_j)^T\mathbf{M}_s(\mathbf{e}_i+\mathbf{e}_j) = 0\ ] \ \ => \ \ [\ \forall\,(i,j),\ M_{ii}+M_{ij}+M_{ji}+M_{ii} = 0\ ]$$

which, using the first result, implies that

$$\forall\,(i,j)\,,\ \ M_{ji} = -\,M_{ij}$$

In our later analysis of linear systems, we will often use quadratic functions of the form $\mathbf{x}^T\mathbf{M}\mathbf{x}$ as Lyapunov function candidates. In view of the above, each quadratic function of this form, whether $\mathbf{M}$ is symmetric or not, is always equal to a quadratic function with a symmetric matrix. Thus, in considering quadratic functions of the form $\mathbf{x}^T\mathbf{M}\mathbf{x}$ as Lyapunov function candidates, one can always assume, without loss of generality, that $\mathbf{M}$ is symmetric.

We are now in a position to introduce the important concept of positive definite matrices.

Definition 3.11 *A square $n \times n$ matrix* **M** *is* positive definite *(p.d.) if*

$$\mathbf{x} \neq \mathbf{0} \quad \Rightarrow \quad \mathbf{x}^T \mathbf{M}\, \mathbf{x} > 0$$

In other words, a matrix **M** is positive definite if the quadratic function $\mathbf{x}^T\mathbf{M}\mathbf{x}$ is a positive definite function. This definition implies that to every positive definite matrix is associated a positive definite function. Obviously, the converse is not true.

Geometrically, the definition of positive-definiteness can be interpreted as simply saying that the angle between a vector **x** and its image **Mx** is always less than 90^o (Figure 3.18).

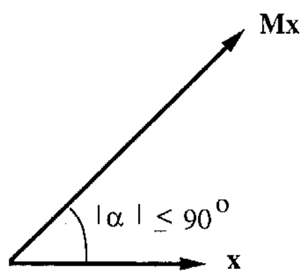

Figure 3.18 : Geometric interpretation of the positive-definiteness of a matrix **M**

A *necessary* condition for a square matrix **M** to be *p.d.* is that its diagonal elements be strictly positive, as can be seen by applying the above definition to the basis vectors. A famous matrix algebra result called Sylvester's theorem shows that, *assuming that* **M** *is symmetric*, a *necessary and sufficient* condition for **M** to be *p.d.* is that its principal minors (*i.e.*, M_{11}, $M_{11}M_{22} - M_{21}M_{12}$, ... , $\det \mathbf{M}$) all be strictly positive; or, equivalently, that all its eigenvalues be strictly positive. In particular, a *p.d.* matrix is always invertible, because the above implies that its determinant is non-zero.

A positive definite matrix **M** can always be decomposed as

$$\mathbf{M} = \mathbf{U}^T \Lambda\, \mathbf{U} \tag{3.17}$$

where **U** is a matrix of eigenvectors and satisfies $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, and Λ is a diagonal matrix containing the eigenvalues of the matrix **M**. Let $\lambda_{min}(\mathbf{M})$ denote the smallest eigenvalue of **M** and $\lambda_{max}(\mathbf{M})$ the largest. Then, it follows from (3.17) that

$$\lambda_{min}(\mathbf{M})\, \|\mathbf{x}\|^2 \leq \mathbf{x}^T\mathbf{M}\mathbf{x} \leq \lambda_{max}(\mathbf{M})\, \|\mathbf{x}\|^2$$

This is due to the following three facts:

- $\mathbf{x}^T\mathbf{M}\,\mathbf{x} = \mathbf{x}^T\mathbf{U}^T\boldsymbol{\Lambda}\,\mathbf{U}\,\mathbf{x} = \mathbf{z}^T\boldsymbol{\Lambda}\,\mathbf{z}$, where $\mathbf{U}\,\mathbf{x} = \mathbf{z}$
- $\lambda_{min}(\mathbf{M})\,\mathbf{I} \leq \boldsymbol{\Lambda} \leq \lambda_{max}(\mathbf{M})\,\mathbf{I}$
- $\mathbf{z}^T\mathbf{z} = \|\mathbf{x}\|^2$

The concepts of positive semi-definite, negative definite, and negative semi-definite can be defined similarly. For instance, a square $n \times n$ matrix $\mathbf{M}$ is said to be *positive semi-definite* (*p.s.d.*) if

$$\forall\,\mathbf{x}\,,\ \mathbf{x}^T\,\mathbf{M}\,\mathbf{x} \geq 0$$

By continuity, necessary and sufficient conditions for positive semi-definiteness are obtained by substituting "positive or zero" to "strictly positive" in the above conditions for positive definiteness. Similarly, a *p.s.d.* matrix is invertible only if it is actually *p.d.* Examples of *p.s.d.* matrices are $n \times n$ matrices of the form $\mathbf{M} = \mathbf{N}^T\mathbf{N}$ where $\mathbf{N}$ is a $m \times n$ matrix. Indeed,

$$\forall\,\mathbf{x}\,,\ \mathbf{x}^T\,\mathbf{N}^T\mathbf{N}\,\mathbf{x} = (\mathbf{N}\mathbf{x})^T\,(\mathbf{N}\mathbf{x}) \geq 0$$

A matrix inequality of the form

$$\mathbf{M}_1 > \mathbf{M}_2$$

(where $\mathbf{M}_1$ and $\mathbf{M}_2$ are square matrices of the same dimension) means that

$$\mathbf{M}_1 - \mathbf{M}_2 > \mathbf{0}$$

i.e., that the matrix $\mathbf{M}_1 - \mathbf{M}_2$ is positive definite. Similar notations apply to the concepts of positive semi-definiteness, negative definiteness, and negative semi-definiteness.

A time-varying matrix $\mathbf{M}(t)$ is *uniformly* positive definite if

$$\exists\,\alpha > 0,\ \forall\,t \geq 0,\ \mathbf{M}(t) \geq \alpha\,\mathbf{I}$$

A similar definition applies for uniform negative-definiteness of a time-varying matrix.

LYAPUNOV FUNCTIONS FOR LINEAR TIME-INVARIANT SYSTEMS

Given a linear system of the form $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, let us consider a quadratic Lyapunov function candidate

$$V = \mathbf{x}^T\,\mathbf{P}\,\mathbf{x}$$

where **P** is a given symmetric positive definite matrix. Differentiating the positive definite function V along the system trajectory yields another quadratic form

$$\dot{V} = \dot{\mathbf{x}}^T \mathbf{P}\,\mathbf{x} + \mathbf{x}^T \mathbf{P}\,\dot{\mathbf{x}} = -\ \mathbf{x}^T \mathbf{Q}\,\mathbf{x} \tag{3.18}$$

where

$$\mathbf{A}^T \mathbf{P} + \mathbf{P}\,\mathbf{A} = -\,\mathbf{Q} \tag{3.19}$$

The question, thus, is to determine whether the symmetric matrix **Q** defined by the so-called *Lyapunov equation* (3.19) above, is itself *p.d.* If this is the case, then V satisfies the conditions of the basic theorem of section 3.4, and the origin is globally asymptotically stable. However, this "natural" approach may lead to inconclusive result, *i.e.*, **Q** may be not positive definite even for stable systems.

Example 3.17: Consider a second-order linear system whose **A** matrix is

$$A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

If we take **P** = **I**, then

$$-Q = \mathbf{P}\,\mathbf{A} + \mathbf{A}^T\mathbf{P} = \begin{bmatrix} 0 & -4 \\ -4 & -24 \end{bmatrix}$$

The matrix **Q** is not positive definite. Therefore, no conclusion can be drawn from the Lyapunov function on whether the system is stable or not. □

A more useful way of studying a given linear system using scalar quadratic functions is, instead, to derive a positive definite matrix **P** from a given positive definite matrix **Q**, *i.e.*,

- choose a positive definite matrix **Q**
- *solve* for **P** from the Lyapunov equation (3.19)
- check whether **P** is *p.d*

If **P** is *p.d.*, then $(1/2)\mathbf{x}^T\mathbf{P}\mathbf{x}$ is a Lyapunov function for the linear system and global asymptotical stability is guaranteed. Unlike the previous approach of going from a given **P** to a matrix **Q**, this technique of going from a given **Q** to a matrix **P** always leads to conclusive results for stable linear systems, as seen from the following theorem.

Theorem 3.6 *A necessary and sufficient condition for a LTI system* $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ *to be strictly stable is that, for any symmetric p.d. matrix* $\mathbf{Q}$*, the unique matrix* $\mathbf{P}$ *solution of the Lyapunov equation (3.19) be symmetric positive definite.*

Proof: The above discussion shows that the condition is sufficient, thus we only need to show that it is also necessary. We first show that given any symmetric *p.d.* matrix $\mathbf{Q}$, there exists a symmetric *p.d.* matrix $\mathbf{P}$ verifying (3.19). We then show that for a given $\mathbf{Q}$, the matrix $\mathbf{P}$ is actually unique.

Let $\mathbf{Q}$ be a given symmetric positive definite matrix, and let

$$\mathbf{P} = \int_0^{\infty} \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,dt \tag{3.20}$$

One can easily show that this integral exists if and only if $\mathbf{A}$ is strictly stable. Also note that the matrix $\mathbf{P}$ thus defined is symmetric and positive definite, since $\mathbf{Q}$ is. Furthermore, we have

$$\begin{aligned} -\mathbf{Q} &= \int_{t=0}^{\infty} d[\,\exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,] \\ &= \int_{t=0}^{\infty} [\,\mathbf{A}^T \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t) + \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,\mathbf{A}\,]\,dt \\ &= \mathbf{A}^T\mathbf{P} + \mathbf{P}\,\mathbf{A} \end{aligned}$$

where the first equality comes from the stability of $\mathbf{A}$ (which implies that $\exp(\mathbf{A}\infty) = \mathbf{0}$), the second from differentiating the exponentials explicitly, and the third from the fact that $\mathbf{A}$ is constant and therefore can be taken out of the integrals.

The uniqueness of $\mathbf{P}$ can be verified similarly by noting that another solution $\mathbf{P}_1$ of the Lyapunov equation would necessarily verify

$$\begin{aligned} \mathbf{P}_1 &= -\int_{t=0}^{\infty} d[\,\exp(\mathbf{A}^T t)\,\mathbf{P}_1\,\exp(\mathbf{A}\,t)\,] \\ &= \int_{t=0}^{\infty} \exp(\mathbf{A}^T t)\,(\,\mathbf{A}^T\mathbf{P}_1 + \mathbf{P}_1\,\mathbf{A}\,)\,\exp(\mathbf{A}\,t)\,]\,dt \\ &= \int_0^{\infty} \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,dt \;=\; \mathbf{P} \end{aligned}$$

An alternate proof of uniqueness is the elegant original proof given by Lyapunov, whic makes direct use of fundamental algebra results. The Lyapunov equation (3.19) can b interpreted as defining a linear map from the n^2 components of the matrix **P** to the n^2 components of the matrix **Q**, where **P** and **Q** are arbitrary (not necessarily symmetric *p.d.*) square matrices. Since (3.20) actually shows the existence of a solution **P** for *any* square matrix **Q**, the range of this linear map is full, and therefore its null-space is reduced to **0**. Thus, for any **Q**, the solution **P** is unique. □

The above theorem shows that *any* positive definite matrix **Q** can be used to determine the stability of a linear system. A simple choice of **Q** is the identity matrix.

Example 3.18: Consider again the second-order system of Example 3.17. Let us take $\mathbf{Q} = \mathbf{I}$ and denote **P** by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where, due to the symmetry of **P**, $p_{21} = p_{12}$. Then the Lyapunov equation is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

whose solution is

$$p_{11} = 5, \; p_{12} = p_{22} = 1$$

The corresponding matrix

$$\mathbf{P} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

is positive definite, and therefore the linear system is globally asymptotically stable. Note that we have solved for **P** directly, without using the more cumbersome expression (3.20). □

Even though the choice $\mathbf{Q} = \mathbf{I}$ is motivated by computational simplicity, it has a surprising property: the resulting Lyapunov analysis allows us to get the best estimate of the state convergence rate, as we shall see in section 3.5.5.

3.5.2 Krasovskii's Method

Let us now come back to the problem of finding Lyapunov functions for general nonlinear systems. Krasovskii's method suggests a simple form of Lyapunov function